

# Pricing Volatility Options under Stochastic Skew with Application to the VIX Index

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## Abstract

In recent years there has been a remarkable growth of volatility options. In particular, VIX options are among the most actively trading contracts at CBOE. These options exhibit upward sloping volatility skew and the shape of the skew is largely independent of the volatility level. To take into account these stylized facts, this article introduces a novel two-factor stochastic volatility model with mean reversion that accounts for stochastic skew consistent with empirical evidence. Importantly, the model is analytically tractable. In this sense, I solve the pricing problem corresponding to standard-start, as well as to forward-start European options through the Fast Fourier Transform.

To illustrate the practical performance of the model, I calibrate the model parameters to the quoted prices of European options on the VIX index. The calibration results are fairly good indicating the ability of the model to capture the shape of the implied volatility skew associated with VIX options.

**Keywords:** volatility options, multifactor stochastic volatility, stochastic skew, mean reversion, forward-start.

**JEL:** G12, G13.

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# 1 Introduction

Volatilities of financial assets are of great importance for derivatives pricing, as well as for portfolio theory. The fact that volatilities are stochastic is widely recognized and in recent years new derivatives, that have some measure of volatility as the underlying asset, are emerging. In this sense, volatility has become an asset class. In 2004 the Chicago Board Options Exchange (CBOE) introduced futures traded on the CBOE Volatility Index (VIX index) and in 2006 options on that index. The VIX index started to be calculated in 1993 and was originally designed to measure the market's expectation of 30-day at-the-money implied volatility associated with the Standard and Poor's 100 index. But with the new methodology<sup>1</sup> implemented in 2003, the squared of the VIX index approximates the variance swap rate or delivery price of a variance swap, obtained from the European options corresponding to the Standard and Poor's 500 index with maturity within one month.

Since the introduction of the VIX index, exchanges in several countries have launched volatility indices. For instance, in 1995 the Deutsche Börse introduced the VDAX index as a measure of the at-the-money implied volatility associated with the DAX equity index. In 2005 the Deutsche Börse launched the VDAX-NEW. This index is calculated with a similar methodology<sup>2</sup> as the current VIX index and represents the 30 day expected market volatility of the DAX index.

The first model to price options on an implied volatility index was originated by Whaley (1993) who applied the Black (1976) valuation formula to price calls on futures contracts. Bali and Ozgur (2008), among others, show that the existence of persistence and mean reversion is quite relevant in stock market volatility. To account for this persistence Grünbichler and Longstaff (1996) use the square root process to model the behavior of a standard deviation index such as the VIX index, whereas Detemple and Osakwe (2000) propose a log-normal Ornstein-Uhlenbeck process.

As pointed out by Sepp (2008) the implied volatility of VIX options displays upward sloping skew. The reason is that out-of-the-money calls on the VIX index offer protection against an equity market crash. Unfortunately, none of the previous models is able to account for this important feature of volatility options. Recently, a number of models have been proposed

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<sup>1</sup>For a definition of the methodology and the history of the VIX index, see CBOE (2009) and Carr and Wu (2006).

<sup>2</sup>See Deutsche Börse AG (2006).

to account for the upward sloping skew associated with the VIX index. These models can be included in two main categories. The first category corresponds to models that impose certain dynamics directly for the volatility index. Within this group we have, among others, the work of Mencía and Sentana (2009) who extend the models of Grünbichler and Longstaff (1996), as well as of Detemple and Osakwe (2000) to incorporate stochastic volatility, jumps and a time varying central tendency in the evolution of the VIX index. They conclude that the introduction of stochastic volatility within the Ornstein-Uhlenbeck specification for the logarithm of the volatility index provides an important improvement in the options pricing performance. Conversely, the introduction of central tendency or jumps has little impact on options on volatility. Unfortunately, calibration errors with respect to market implied volatilities are remarkably high making the model hardly to use for trading purposes.

The second category corresponds to those models where the volatility index is derived from the assumed returns dynamics. Among others, Sepp (2008), Gatheral (2008), who considers stochastic volatility and central tendency, and Lin and Chang (2009, 2010) follow this approach. Unfortunately, as shown by Cheng et al. (2012), Lin and Chang's formula for volatility derivatives is not an exact solution of their pricing equation and their formula cannot serve as a reasonable approximation. On the other hand, Bergomi (2005) models the joint dynamics of forward variance swap rates and the underlying return process but, as explained by Gatheral (2008), this model generates almost no skew for VIX options.

Although from a theoretical perspective is quite interesting to consider the relationship between the volatility index and the underlying equity index, from a practical point of view, since hedging of VIX options is typically done with trading in futures contracts, the valuation of volatility options can be addressed postulating directly the dynamics associated with the volatility process. This situation has certain analogy with the valuation of options on equity indices. The return on the index is the weighted average of the returns on the individual assets but the dynamics of index returns are not so simple. For instance, in the Black-Scholes (1973) context the assumption that the underlying asset for an option follows a geometric Brownian motion is convenient for individual stocks, but a linear combination of log-normal variables does not have a log-normal distribution. Nevertheless, the standard practice, in the literature and among practitioners, is to set directly the dynamics associated with the evolution of the index without considering the dynamics corresponding to its constituents.

Wang and Daigler (2011) examine the pricing performance of several VIX option models

including Whaley (1993), Grünbichler and Longstaff (1996) and Lin and Chang (2009), using actual VIX option market prices. One of the conclusions of this study is that an adequate pricing model has yet to be developed. In this sense, Cheng et al. (2012) postulate that a commonly accepted VIX option pricing model is not available yet.

As documented in this article, another important feature of VIX options is that there is considerable time-variation in the implied volatility skew. This article contributes to the literature presenting a valuation model that is consistent with this stylized fact. To this end, I follow the first approach and I introduce a two-factor stochastic volatility model within the Ornstein-Uhlenbeck specification for the logarithm of the volatility index of Detemple and Osakwe (2000) to price options on volatility. This novel valuation model accounts for mean reversion, stochastic volatility and stochastic correlation between the volatility index and its instantaneous stochastic variance. Hence, the model is able to generate stochastic implied volatility skew consistent with empirical evidence. Importantly, the model is analytically tractable. In this sense, I solve the pricing problem corresponding to European options through the Fast Fourier Transform as in Carr and Madan (1999).

The existence of stochastic skew has been documented by Derman (1999) and Christoffersen et al. (2009) in the equity context, as well as by Carr and Wu (2007) for foreign exchange options markets. Christoffersen et al. (2009), as well as da Fonseca et al. (2008) propose multifactor stochastic volatility models to account for the existence of stochastic skew in the valuation of equity options. But, to the best of my knowledge, the model presented in this paper represents the first multifactor stochastic volatility model to price volatility options under the existence of mean reversion in the underlying volatility index.

Forward-start options, i.e. options with a strike price that will be determined at a later date are usually embedded in equity structured products. Although, nowadays, forward-start options on volatility are not common, it is likely that they experience an upswing with the future development of volatility options markets. Hence, another contribution of this paper consists of providing semi-closed-form solutions for the price of forward-start volatility options under the model presented in this article.

The rest of the paper is organized as follows. Section 2 analyzes the dynamics associated with the implied volatility surface corresponding to the VIX volatility index and describes the main features of VIX options. Section 3 presents the multifactor stochastic volatility model with mean reversion. Section 4 considers the pricing problem and provides semi-closed-form

solutions for the price of standard-start, as well as forward-start European options on volatility. Section 5 shows an example of its implementation as applied to the VIX index options market. Finally, section 6 concludes.

## 2 Empirical properties of volatility options

As said previously, exchanges in several countries have launched volatility indices but the VIX index has effectively become the standard measure of volatility risk between investors and it is the only volatility index with listed options. They are European-style options with cash settlement according to the difference between the value of the VIX at expiration and the strike price. Note that these options can also be interpreted as options on VIX futures with the same maturity as the options. This fact can be used to simplify the pricing problem.

Interestingly, VIX options are among the most actively trading contracts at CBOE. One of the reasons is that they allow investors to be hedged against the equity market downside. Panel A of figure 1 displays the weekly evolution of the at-the-money implied volatility associated with options with maturity within three months during the period January 6, 2010 to April 18, 2012. On the other hand, panel B shows the evolution of the 95-100 skew, defined as  $\Sigma_1 - \Sigma_{0.95}$ , as well as the 100-105 skew, defined as  $\Sigma_{1.05} - \Sigma_1$ , where  $\Sigma_K$  is the implied volatility corresponding to options with strike equal to  $K$  expressed as a percentage of the underlying asset. The data have been obtained from Bloomberg. We can see from the figure that both measures of skew are always positive indicating the existence of an upward sloping volatility skew. The figure also shows that the at-the-money implied volatility, as well as the skew evolves stochastically through time. Moreover, it seems that the shape of the skew is largely independent of the volatility level. To corroborate this fact, panel A of figure 2 provides a scatter plot, which gives the relation between the 95-100 skew and the at-the-money implied volatility, while panel B displays the scatter plot associated with the 100-105 skew. The figure shows that there are low volatility days with a steep volatility slope, as well as a flat volatility slope. On the other hand, we also have high volatility days with steep and flat volatility slopes<sup>3</sup>. This fact was first noticed by Derman (1999) in the context of the implied volatility corresponding to equity indices and it have been also documented by Christoffersen et al. (2009) for the implied volatility associated with the Standard and Poor's 500 equity index.

Importantly, in single-factor stochastic volatility models the correlation between the volatil-

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<sup>3</sup>The same behavior is observed for implied volatilities associated with other maturities.

ity index and its instantaneous variance is constant. This structural limitation does not allow this kind of models to capture the time-varying nature of the volatility skew. To account for this stylized fact, the following section introduces a two-factor stochastic volatility model with mean reversion that have more flexibility to model the level and slope of the skew, as well as the volatility term structure.

### 3 A two-factor stochastic volatility model

To take into account the mean reversion observed in volatility indices, such as the VIX index, and the time-varying nature of the volatility skew observed in volatility options, this section introduces the two-factor stochastic volatility model and describes its variance-covariance structure. Let  $X_t$  denote the spot price of the underlying volatility process at time  $t \in [0, \Upsilon]$ , where  $\Upsilon$  is some arbitrarily distant horizon and let us denote by  $Y_t = \ln X_t$  the log-return process. For simplicity, I assume that the continuously compounded risk-free rate  $r$  is constant. Let  $\Theta$  denote the probability measure defined on a probability space  $(\Omega, \mathcal{F}, \Theta)$  such that asset prices expressed in terms of the current account are martingales. We denote this probability measure as the risk-neutral measure. I assume the following dynamics for the log-return process  $Y_t$  under  $\Theta$ :

$$dY_t = \kappa_Y (\theta_Y - Y_t) dt + \sum_{i=1}^2 \sqrt{v_{it}} dZ_{it}$$

with:

$$dv_{it} = \kappa_i (\theta_i - v_{it}) dt + \sigma_i \sqrt{v_{it}} dW_{it}$$

where  $\theta_Y$  is the long-term mean associated with the log-return process and  $\kappa_Y$  denotes the speed of mean reversion. Analogously,  $\theta_i$  represents the long-term mean corresponding to the instantaneous variance factor  $i$  (for  $i = 1, 2$ ),  $\kappa_i$  denotes the speed of mean reversion and, finally,  $\sigma_i$  represents the volatility of the variance factor  $i$ . For analytical convenience, let us rewrite the previous equations as follows:

$$dY_t = (a_Y - b_Y Y_t) dt + \sum_{i=1}^2 \sqrt{v_{it}} dZ_{it} \tag{1}$$

$$dv_{it} = (a_i - b_i v_{it}) dt + \sigma_i \sqrt{v_{it}} dW_{it} \tag{2}$$

where  $b_Y = \kappa_Y$ ,  $a_Y = \kappa_Y \theta_Y$ ,  $b_i = \kappa_i$  and  $a_i = \kappa_i \theta_i$ . In equations (1) and (2)  $Z_{it}$  and  $W_{it}$  are Wiener processes such that:

$$dZ_{it}dW_{jt} = \begin{cases} \rho_i dt & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

On the other hand,  $Z_{1t}$  and  $Z_{2t}$  are uncorrelated. In addition,  $W_{1t}$  and  $W_{2t}$  are also uncorrelated. I denote the specification of equations (1) and (2) the two-factor stochastic volatility with mean reversion (TFSV-MR) model.

### 3.1 Variance-covariance structure

The conditional variance of the return process is:

$$V_t := \frac{1}{dt} \text{Var}(dY_t) = \sum_{i=1}^2 v_{it}$$

whereas the variance associated with  $dV_t$  is given by:

$$\Pi_t := \frac{1}{dt} \text{Var}(dV_t) = \sum_{i=1}^2 \sigma_i^2 v_{it}$$

Both variance processes are stochastic as in single-factor volatility models but the multifactor specification provides more flexibility to model the term structure of volatility. Moreover, under the multifactor volatility model the correlation between the log-return corresponding to the volatility index and its instantaneous variance is also stochastic and it can be expressed as:

$$\rho_{X_t V_t} := \text{Corr}(dY_t, dV_t) = \frac{\rho_1 \sigma_1 v_{1t} + \rho_2 \sigma_2 v_{2t}}{\sqrt{\sigma_1^2 v_{1t} + \sigma_2^2 v_{2t}} \sqrt{v_{1t} + v_{2t}}}$$

Importantly, the consideration of two stochastic volatility factors potentially enables the model to capture the time-varying behavior of the skew. Moreover, it provides more flexibility to model term structure effects. Note that if, in the previous expression, we drop factor 2 we have that the correlation between the asset return and the variance process simplifies to the constant correlation level  $\rho_1$ . Hence, single-factor volatility specifications have a structural limitation to account for the existence of stochastic skew.

## 4 The pricing problem

In this section I follow the methodology of Carr and Madan (1999) to calculate option prices efficiently in terms of the Fast Fourier Transform. In this sense, let us consider a generic payoff on the terminal value of the underlying asset, at time  $t = T$ , under the risk-neutral probability measure  $w(Y_T)$ . From the Fundamental Theorem of Asset Pricing we have that the time  $t = 0$  price of this option, denoted  $OP_0$ , is given by:

$$OP_0 = e^{-rT} E_{\Theta} [w(Y_T)] = e^{-rT} \int_{\mathbb{R}} w(Y_T) \delta_T(Y_T) dY_T$$

where  $\delta_T(Y_T)$  is the risk-neutral density function of  $Y_T$ . In the particular case of a European call with strike  $K$ , the payoff function becomes:

$$w(Y_T) := (X_T - K)^+ = (e^{Y_T} - e^{\ln K})^+$$

Carr and Madan (1999) show that the time  $t = 0$  price associated with the European call with maturity  $t = T$  and strike price  $K$  can be expressed as follows:

$$C(X_0, T, K) = \frac{e^{-(rT + \alpha \ln K)}}{\pi} \int_0^{\infty} e^{-iz \ln K} \frac{\psi(z - i(1 + \alpha); Y_0, T)}{(\alpha + 1 + iz)(\alpha + iz)} dz \quad (3)$$

where the parameter  $\alpha$  is introduced in order to have an integrable function,  $i^2 = -1$  and where  $\psi(u; Y_0, T)$  is the characteristic function associated with the asset returns defined as:

$$\psi(u; Y_0, T) := E_{\Theta} [e^{iuY_T}] = \int_{\mathbb{R}} e^{iuY_T} \delta_T(Y_T) dY_T \quad (4)$$

Hence, the call option price is known once the parameter  $\alpha$  is chosen<sup>4</sup> and the characteristic function is found explicitly, which is the case of the TFSV-MR model.

### 4.1 The characteristic function

Appendix A shows that, under the risk-neutral measure  $\Theta$ , the characteristic function associated with the TFSV-MR model  $\psi(u; Y_0, T)$  is given by:

$$\psi(u; Y_0, T) = e^{B(\lambda, T) + \sum_{i=1}^2 A_i(\lambda, T) v_{i0} + c(\lambda, T) Y_0} \quad (5)$$

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<sup>4</sup>In my experience  $\alpha = 1.25$  provides good results.



with  $\lambda = iu$ ,  $c(\lambda, T) = \lambda e^{-Tb_Y}$  and where:

$$\frac{\partial A_i(\lambda, t)}{\partial t} = \frac{\lambda^2 e^{-2tb_Y}}{2} + A_i(\lambda, t) \left[ \rho_i \sigma_i \lambda e^{-tb_Y} - b_i \right] + \frac{\sigma_i^2}{2} A_i^2(\lambda, t) \quad \text{for } i = 1, 2 \quad (6)$$

with boundary conditions  $A_i(\lambda, 0) = 0$ . On the other hand,  $B(\lambda, T)$  can be obtained as follows:

$$B(\lambda, T) = \frac{\lambda a_Y}{b_Y} \left[ 1 - e^{-Tb_Y} \right] + \sum_{i=1}^2 a_i \int_0^T A_i(\lambda, t) dt$$

In general, the Riccati equations (6) do not admit closed-form solutions. However, they can be solved numerically very efficiently using a Runge-Kutta algorithm<sup>5</sup>. In the particular case where  $b_Y = b_1 = b_2 = b$ , that is, when the speeds of mean reversion associated with the logarithm of the volatility index, as well as with the variance factors coincide, then we have the following closed-form expression corresponding to  $A_i(\lambda, T)$ :

$$A_i(\lambda, T) = \frac{\lambda^2 e^{-Tb} \left[ e^{g_i(T)} - 1 \right]}{\rho_i \sigma_i \lambda \left[ 1 - e^{g_i(T)} \right] + \left[ 1 + e^{g_i(T)} \right] \sqrt{\sigma_i^2 \lambda^2 (\rho_i^2 - 1)}} \quad \text{for } i = 1, 2$$

$$g_i(T) = \frac{\left( 1 - e^{-Tb} \right)}{b} \sqrt{\sigma_i^2 \lambda^2 (\rho_i^2 - 1)}$$

Although the assumption that leads to this closed-form solution can be quite restrictive, it is useful to check the accuracy of the numerical method, as well as to obtain very quickly relatively good starting points for the calibration procedure associated with the unconstrained specification.

Note that we can combine equation (3) together with equation (5) to obtain a semi-closed-form solution for the price of a European call under the assumptions of the TFSV-MR model. On the other hand, the time  $t = 0$  value corresponding to a forward contract with maturity  $t = T$  on the volatility index can be expressed as:

$$F_{0,T}(X_T) := E_{\Theta} [X_T] = E_{\Theta} \left[ e^{Y_T} \right] = \psi(-i; Y_0, T)$$

From the put-call parity it is easy to obtain the price associated with a European put combining the pricing formula corresponding to the call and the explicit solution for the forward price of the previous equation.

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<sup>5</sup>This method is also used by da Fonseca and Grasselli (2011) to compute the characteristic function of equity assets returns under multifactor stochastic specifications.

## 4.2 Pricing forward-start options

Sometimes investors can be interested in European options on a volatility index with a strike price that will be determined a later date. These forward start options are quite sensitive to the evolution of the volatility of volatility. This section considers the pricing problem associated with forward-start European options under the TFSV-MR model. To this end, I define the forward log-return process as  $Y_{t,T} := Y_T - Y_t$ . Let us consider a forward-start European call that depends on the evolution of the asset price between the strike date  $t$  and the maturity date  $T$  with payoff:

$$w(Y_{t,T}) := \left( \frac{X_T}{X_t} - K \right)^+ = \left( e^{Y_T - Y_t} - e^{\ln K} \right)^+$$

The time  $t = 0$  price corresponding to the forward-start European call under the risk-neutral probability measure  $\Theta$ ,  $C_0(X_0, t, T, K)$ , can be expressed as follows:

$$C_0(X_0, t, T, K) = e^{-rT} E_{\Theta} \left[ \left( e^{Y_T - Y_t} - e^{\ln K} \right)^+ \right]$$

Let us denote by  $\psi_Y(u; t, T)$  the characteristic function associated with the forward log-return process  $Y_T - Y_t$ , denoted forward characteristic function:

$$\psi_Y(u; t, T) := E_{\Theta} \left[ e^{iu(Y_T - Y_t)} \right]$$

It is possible to express the previous equation as follows:

$$\psi_Y(u; t, T) = E_{\Theta} \left[ e^{-iuY_t} E_{t, \Theta} \left[ e^{iuY_T} \right] \right]$$

where  $E_{t, \Theta}[\cdot]$  denotes the expectation conditional on the information available through time  $t$ .

Taking into account equations (4) and (5), we can rewrite the previous expression as:

$$\psi_Y(u; t, T) = E_{\Theta} \left[ e^{-iuY_t} \psi(u; Y_t, T - t) \right] = e^{B(\lambda, T-t)} E_{\Theta} \left[ e^{\sum_{i=1}^2 A_i(\lambda, T-t) v_{it} + [c(\lambda, T-t) - \lambda] Y_t} \right]$$

Let  $\Psi(H_i, Q; V_0, Y_0, t)$  denote the Laplace transform corresponding to the variance process  $V_t$  and the log-return process  $Y_t$  under the risk-neutral probability measure  $\Theta$  defined as:

$$\Psi(H_i, Q; V_0, Y_0, t) := E_{\Theta} \left[ e^{QY_t + \sum_{i=1}^2 H_i v_{it}} \right] \quad H_i, Q \in \mathbb{R} \quad \text{for } i = 1, 2$$

If we set  $H_i = A_i(\lambda, T - t)$  and  $Q = c(\lambda, T - t) - \lambda$ , appendix B shows that the Laplace transform  $\Psi(A_i(\lambda, T - t), c(\lambda, T - t) - \lambda; V_0, Y_0, t)$  is given by:

$$\Psi(A_i(\lambda, T - t), c(\lambda, T - t) - \lambda; V_0, Y_0, t) = e^{L(t) + \sum_{i=1}^2 R_i(t)v_{i0} + M(t)Y_0}$$

where

$$M(t) = Qe^{-tb_Y}$$

$$\frac{\partial R_i(t)}{\partial t} = \frac{Q^2 e^{-2tb_Y}}{2} + R_i(t) [\rho_i \sigma_i Q e^{-tb_Y} - b_i] + \frac{\sigma_i^2}{2} R_i^2(t) \quad \text{for } i = 1, 2 \quad (7)$$

with boundary conditions  $R_i(0) = H_i$  and where  $L(t)$  can be obtained as follows:

$$L(t) = \frac{Qa_Y}{b_Y} [1 - e^{-tb_Y}] + \sum_{i=1}^2 a_i \int_0^t R_i(s) ds$$

Therefore, the forward characteristic function is given by:

$$\psi_Y(u; t, T) := e^{B(\lambda, T-t) + L(t) + \sum_{i=1}^2 R_i(t)v_{i0} + M(t)Y_0} \quad (8)$$

Hence, it is possible to use once again the Fast Fourier transform to express the time  $t = 0$  price associated with a forward-start European call on the volatility index as follows:

$$C_0(X_0, t, T, K) = \frac{e^{-(rT + \alpha \ln K)}}{\pi} \int_0^\infty e^{-iz \ln K} \frac{\psi_Y(z - i(1 + \alpha); t, T)}{(\alpha + 1 + iz)(\alpha + iz)} dz \quad (9)$$

where the forward characteristic function is given by equation (8). Importantly, in the context of equity options, da Fonseca et al. (2008) show that the value of forward-start options is independent on the level of the underlying asset and depends only on the volatility process. But, under the existence of mean reversion in the evolution of the underlying asset, the value of forward-start options is also affected by the underlying asset price.

As in the case of standard-start options, in general, the differential equations (7) do not admit closed-form solutions but can be easily solved using Runge-Kutta methods. As before, if

we assume that  $b_Y = b_1 = b_2 = b$ , then we can obtain  $R_i(t)$  in closed-form:

$$R_i(t) = e^{-tb} \left[ \frac{Q^2 [e^{f_i(t)} - 1] + H_i \left[ (e^{f_i(t)} - 1) G_i + (1 + e^{f_i(t)}) \sqrt{G_i^2 - Q^2 \sigma_i^2} \right]}{(1 - e^{f_i(t)}) [G_i + H_i \sigma_i^2] + (1 + e^{f_i(t)}) \sqrt{G_i^2 - Q^2 \sigma_i^2}} \right] \quad \text{for } i = 1, 2$$

$$f_i(t) = \frac{(1 - e^{-tb})}{b} \sqrt{G_i^2 - Q^2 \sigma_i^2}$$

$$G_i = \rho_i \sigma_i Q$$

## 5 Illustration

### 5.1 Calibration to market data

In this section I analyze the practical ability of the TFSV-MR model to replicate the quoted prices of European options. When we calibrate a model, we have to specify if we choose a set of options quoted at a fixed day or a times series of option prices. The market practice is to perform a calibration per day. Bakshi et al. (1997), Carr et al. (2003) and da Fonseca and Grasselli (2011), among others, follow this approach. In fact, as explained by da Fonseca and Grasselli (2011), if we perform a calibration on a time series of option prices, since the volatility is not observable, it would have to be considered as a parameter and then estimated together with the other parameters. But this strategy leads to optimize a function with respect to a large number of variables which can become too difficult numerically and can give odd solutions. Hence, it is desirable that the calibration process involves the optimization of a function that should be as simple as possible. In this sense, to illustrate the TFSV-MR model, I calibrate it to VIX options data corresponding to February 22, 2012, obtained from Bloomberg. The data consist of implied volatilities associated with European options with maturities within two months, three months and six months and with moneyness<sup>6</sup> ranging from 80% to 120%. The corresponding reference spot price for the VIX index is 18.19. Figure 3 displays the data graphically. We can see from the figure that in reverse to the downward sloping implied volatility skew associated with equity options, the skew observed in the VIX call options has an upward sloping skew. The figure also displays negative term structure with higher implied volatilities for short-term options. Moreover, for these options the slope of the skew is more pronounced.

Another important question when calibrating a model is related to the choice of the penalizing function. In fact, Christoffersen and Jacobs (2004) show that the choice of the distance

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<sup>6</sup>The moneyness is defined as  $K/X$  where  $K$  is the strike price and  $X$  is the level associated with the VIX index.

has a relevant impact on the calibrated parameters. To improve the calibration of short term implied volatility the market practice consists of putting more weight on short maturity options using the inverse of the vega. In this study, I follow this approach<sup>7</sup> and I choose the model parameters that solve the following optimization problem:

$$\min_{\gamma} \frac{1}{N_i N_j} \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \left[ \frac{C^{mkt}(K_i, T_j) - C_{\gamma}(K_i, T_j)}{\varsigma(K_i, T_j)} \right]^2$$

where  $\gamma$  is the vector of parameters to be estimated,  $C^{mkt}(K_i, T_j)$  is the market price of a European call with strike price  $K_i$  and maturity  $T_j$ ,  $C_{\gamma}(K_i, T_j)$  is the model price,  $\varsigma(K_i, T_j)$  is the vega corresponding to a European option with strike  $K_i$  and maturity  $T_j$ ,  $N_i$  is the total number of strikes and, finally,  $N_j$  represents the number of maturities considered.

Table 1 displays the calibrated parameters values and the mean absolute errors (MAE) associated with implied volatilities, as well as with options prices. In the case of the MAE corresponding to option prices, the table provides the percentage MAE, expressed as a percentage of the VIX index level and the MAE expressed in dollars. The calibration results are fairly good. In particular, the MAE corresponding to implied volatilities is of the same order of magnitude as the calibration errors obtained by Christoffersen et al. (2009) in the estimation of their two-factor stochastic volatility model using options data for the Standard and Poor's 500 equity index. Importantly, since implied volatilities associated with the VIX index are much higher than the implied volatilities corresponding to the Standard and Poor's 500 index, the volatility errors, as a percentage of the market implied volatility, obtained under the TFSV-MR model are considerably lower than the ones obtained under the specification of Christoffersen et al. (2009). Moreover, the dollar errors are much lower than the ones obtained by Wang and Daigler (2011) and Mencía and Sentana (2009) corresponding to different VIX pricing specifications.

It is important to note that, unlike what happens in the case of equity assets, the coefficients  $\rho_i$  are positive to capture the upward sloping skew associated with the implied volatility corresponding to the VIX index.

Panels A, B and C of figure 4 show, respectively, the market implied volatility skew, as well as the one generated by the parameters of table 1 under the TFSV-MR specification corresponding to options with maturity within two months, three months and six months. Finally, panel D of figure 4, displays the calibrated implied volatility surface. We can see from the figure that this volatility surface exhibits positive skew and negative term structure consistent with the

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<sup>7</sup>Christoffersen et al. (2009) and Fonseca and Grasselli (2011), among others, also choose this methodology.

empirical evidence.

One prominent feature of the TFSV-MR model is the two-regime property of the calibrated dynamics. In this sense, we have a low mean reversion, as well as a high correlation and high volatility of volatility regime, which can be associated with the short term skew. On the other hand, we have a quite high mean reversion, as well as a low correlation together with a low volatility of volatility that can be associated with the long term smile. This is an important advantage of the multifactor specification when compared with single-factor volatility models. In this sense, I also calibrated a simplified version of the model with only one factor. In this case, the estimated correlation between the volatility index and the variance factor is equal to 1 and, hence, the dynamics is degenerate indicating that a single factor specification is not flexible enough to capture the shape corresponding to the implied volatility surface associated with the VIX index.

Interestingly, the estimated parameters of table 1 exhibit some striking similarities with the results of Christoffersen et al. (2009). In particular, in both cases, the factor with the highest mean reversion has a smaller volatility of volatility and a smaller correlation parameter (in absolute value). Note that the Feller's condition<sup>8</sup>, which ensures that the variance factors do not reach zero, is not satisfied. Similar results have been obtained, among others, by Christoffersen et al. (2009) and da Fonseca and Grasselli (2011) in the calibration of the Heston (1993) model and other multifactor stochastic volatility models using equity options data.

Importantly, once we have determined the characteristic function associated with the asset returns, we can use the Fourier theorem to obtain the corresponding risk-neutral density function<sup>9</sup>:

$$\delta_T(Y_T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuY_T} \psi(u; Y_0, T) du$$

$$\delta_T^X(X_T) = \frac{\delta_T(Y_T)}{X_T}$$

where  $\delta_T^X(X_T)$  is the risk-neutral density function of the volatility index  $X_T$ . As an illustration, figure 5 shows the calibrated risk-neutral density function corresponding to options with maturity within six months. The figure reveals that the market curve has excess probability in the right tail of the distribution, which is consistent with the existence of an upward sloping

<sup>8</sup>The Feller's condition is given by  $2\kappa_i\theta_i > \sigma_i^2$ .

<sup>9</sup>This approach is equivalent to the result obtained by Breeden and Litzenberger (1978) to calculate the risk-neutral density function in terms of European options prices.

volatility skew.

## 5.2 Forward-start implied volatility skew

As said previously, forward-start options are quite sensitive to the evolution of the volatility of volatility. Since we have available semi-closed-form solutions corresponding to the price of these options under the TFSV-MR model, we can use the calibrated parameters of table 1 to obtain the forward-start volatility skew consistent with the market price of European options under the TFSV-MR specification. In this sense, figure 6 displays the three months forward implied volatility surface. The figure provides the implied volatility of forward-start calls with maturity equal to three months for different start dates and strikes. For each strike  $K_i$ , expressed in percentage terms, and time  $t_j$ , expressed in years, the figure shows the forward implied volatilities associated with the following forward-start calls:

$$C_0 \left( X_0, t_j - \frac{1}{4}, t_j, K_i \right)$$

where  $X_0 = 18.19$  is the spot price corresponding to the VIX index. The figure shows that forward skews are more convex than today's skew. This effect is consistent with the results obtained under other stochastic volatility specifications in the equity context. As pointed out by Bergomi (2004), since the price of a call option is an increasing and convex function of its implied volatility, uncertainty in the value of future implied volatility increases the price of the option.

## 6 Conclusion

In recent years there has been a remarkable growth of volatility options. In particular, VIX options are among the most actively trading contracts at CBOE. These options exhibit upward sloping volatility skew because out-of-the-money calls on the VIX index offer protection against an equity market crash. Another remarkable feature is that the shape of the skew is largely independent of the volatility level. There are low volatility days with a steep volatility slope, as well as a flat volatility slope. On the other hand, we also have high volatility days with steep and flat volatility slopes. Since, in single-factor stochastic volatility models the correlation between the volatility index and its instantaneous variance is constant, they are not able to capture the time-varying nature of the volatility skew.

To take into account these stylized facts, this article introduces a novel two-factor stochas-

tic volatility model with mean reversion that accounts for stochastic correlation between the volatility index and its instantaneous stochastic variance. Hence, the model is able to generate stochastic implied volatility skew consistent with empirical evidence. Importantly, the model is analytically tractable. In this sense, I solve the pricing problem corresponding to standard-start, as well as to forward-start European options through the Fast Fourier Transform.

To analyze the practical performance of the TFSV-MR model to replicate market prices, I calibrate the model parameters to the quoted prices of European options on the VIX index. The calibration results are fairly good indicating the ability of the model to capture the shape of the implied volatility skew associated with VIX options.

## A The characteristic function of the asset returns

As said previously, the characteristic function of the asset returns is given by the following expression:

$$\psi(u; Y_t, T - t) := E_{t, \Theta} \left[ e^{iuY_T} \right]$$

where  $E_{t, \Theta}[\cdot]$  denotes the expectation conditional on the information available through time  $t$ .

From the Feynman-Kac theorem we have that:

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial Y_t} E_{t, \Theta} [dY_t] + \sum_{i=1}^2 \frac{\partial \psi}{\partial v_{it}} E_{t, \Theta} [dv_{it}] + \frac{1}{2} \frac{\partial^2 \psi}{\partial Y_t^2} Var_{t, \Theta} [dY_t] \\ + \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2 \psi}{\partial v_{it}^2} Var_{t, \Theta} [dv_{it}] + \sum_{i=1}^2 \frac{\partial^2 \psi}{\partial Y_t \partial v_{it}} Cov_{t, \Theta} [dY_t, dv_{it}] = 0 \end{aligned}$$

Substituting the expressions corresponding to the expectations and covariances in the previous equation yields:

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial Y_t} [a_Y - b_Y Y_t] + \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2 \psi}{\partial Y_t^2} v_{it} + \sum_{i=1}^2 \frac{\partial \psi}{\partial v_{it}} [a_i - b_i v_{it}] \\ + \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2 \psi}{\partial v_{it}^2} \sigma_i^2 v_{it} + \sum_{i=1}^2 \frac{\partial^2 \psi}{\partial Y_t \partial v_{it}} \sigma_i \rho_i v_{it} = 0 \end{aligned} \quad (10)$$

I postulate the following guess solution:

$$\psi(\lambda; Y_t, T - t) = \psi(\lambda; Y_t, \tau) = e^{B(\lambda, \tau) + \sum_{i=1}^2 A_i(\lambda, \tau) v_{it} + c(\lambda, \tau) Y_t}$$



where  $\lambda = iu$  and  $\tau = T - t$ . The boundary conditions associated with the previous solution are:

$$A_i(\lambda, 0) = 0 \text{ for } i = 1, 2.$$

$$B(\lambda, 0) = 0$$

$$c(\lambda, 0) = \lambda$$

Substituting the guess solution in equation (10) and dividing by  $\psi$  yields:

$$\begin{aligned} & -\frac{\partial B}{\partial \tau} - \sum_{i=1}^2 \frac{\partial A_i}{\partial \tau} v_{it} - \frac{\partial c}{\partial \tau} Y_t + c[a_Y - b_Y Y_t] + \frac{c^2}{2} \sum_{i=1}^2 v_{it} \\ & + \sum_{i=1}^2 A_i [a_i - b_i v_{it}] + \frac{1}{2} \sum_{i=1}^2 A_i^2 \sigma_i^2 v_{it} + c \sum_{i=1}^2 A_i \sigma_i \rho_i v_{it} = 0 \end{aligned}$$

This equation has to hold for all values of  $v_{it}$  and  $Y_t$ . Therefore, we have:

$$\begin{aligned} & -\frac{\partial c}{\partial \tau} - cb_Y = 0 \tag{11} \\ & -\frac{\partial A_i}{\partial \tau} + \frac{c^2}{2} + [c\sigma_i \rho_i - b_i] A_i + \frac{\sigma_i^2}{2} A_i^2 = 0 \quad \text{for } i = 1, 2 \\ & -\frac{\partial B}{\partial \tau} + ca_Y + \sum_{i=1}^2 a_i A_i = 0 \end{aligned}$$

The solution to equation (11) is given by  $c(\lambda, \tau) = \lambda e^{-\tau b_Y}$ . Hence, we have:

$$\frac{\partial A_i(\lambda, \tau)}{\partial \tau} = \frac{\lambda^2 e^{-2\tau b_Y}}{2} + A_i(\lambda, \tau) [\rho_i \sigma_i \lambda e^{-\tau b_Y} - b_i] + \frac{\sigma_i^2}{2} A_i^2(\lambda, \tau) \quad \text{for } i = 1, 2$$

$$B(\lambda, \tau) = \frac{\lambda a_Y}{b_Y} [1 - e^{-\tau b_Y}] + \sum_{i=1}^2 a_i \int_0^\tau A_i(\lambda, s) ds$$

## B The Laplace transform of the variance process and the asset returns

Let us consider the conditional Laplace transform corresponding to the variance process  $V_t$  and the log-return vector  $Y_t$ , under the risk-neutral probability measure  $\Theta$ :

$$\Psi(H_i, Q; V_t, Y_t, T - t) := E_{t, \Theta} \left[ e^{QY_T + \sum_{i=1}^2 H_i v_{iT}} \right] \quad H_i, Q \in \mathbb{R} \text{ for } i = 1, 2$$

From the Feynman-Kac theorem we have that:

$$\begin{aligned} \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial Y_t} [a_Y - b_Y Y_t] + \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2 \Psi}{\partial Y_t^2} v_{it} + \sum_{i=1}^2 \frac{\partial \Psi}{\partial v_{it}} [a_i - b_i v_{it}] \\ + \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2 \Psi}{\partial v_{it}^2} \sigma_i^2 v_{it} + \sum_{i=1}^2 \frac{\partial^2 \Psi}{\partial Y_t \partial v_{it}} \sigma_i \rho_i v_{it} = 0 \end{aligned} \quad (12)$$

I postulate the following guess solution:

$$\Psi(H_i, Q; V_t, Y_t, T-t) = \Psi(H_i, Q; V_t, Y_t, \tau) = e^{L(\tau) + \sum_{i=1}^2 R_i(\tau) v_{it} + M(\tau) Y_t}$$

The boundary conditions associated with the previous solution are:

$$\begin{aligned} R_i(0) &= H_i \text{ for } i = 1, 2. \\ M(0) &= Q \\ L(0) &= 0 \end{aligned}$$

Substituting the guess solution in equation (12) and dividing by  $\Psi$  yields:

$$\begin{aligned} -\frac{\partial L}{\partial \tau} - \sum_{i=1}^2 \frac{\partial R_i}{\partial \tau} v_{it} - \frac{\partial M}{\partial \tau} Y_t + M [a_Y - b_Y Y_t] + \frac{M^2}{2} \sum_{i=1}^2 v_{it} \\ + \sum_{i=1}^2 R_i [a_i - b_i v_{it}] + \frac{1}{2} \sum_{i=1}^2 R_i^2 \sigma_i^2 v_{it} + M \sum_{i=1}^2 R_i \sigma_i \rho_i v_{it} = 0 \end{aligned}$$

This equation has to hold for all values of  $v_{it}$  and  $Y_t$ . Therefore, we have:

$$\begin{aligned} -\frac{\partial M}{\partial \tau} - M b_Y &= 0 \\ -\frac{\partial R_i}{\partial \tau} + \frac{M^2}{2} + [M \sigma_i \rho_i - b_i] R_i + \frac{\sigma_i^2}{2} R_i^2 &= 0 \text{ for } i = 1, 2 \\ -\frac{\partial L}{\partial \tau} + M a_Y + \sum_{i=1}^2 a_i R_i &= 0 \end{aligned} \quad (13)$$

The solution to equation (13) is given by  $M(\tau) = Q e^{-\tau b_Y}$ . Hence, we have:

$$\frac{\partial R_i(\tau)}{\partial \tau} = \frac{Q^2 e^{-2\tau b_Y}}{2} + R_i(\tau) [\rho_i \sigma_i Q e^{-\tau b_Y} - b_i] + \frac{\sigma_i^2}{2} R_i^2(\tau) \text{ for } i = 1, 2$$

$$L(\tau) = \frac{Q a_Y}{b_Y} [1 - e^{-\tau b_Y}] + \sum_{i=1}^2 a_i \int_0^\tau R_i(s) ds$$

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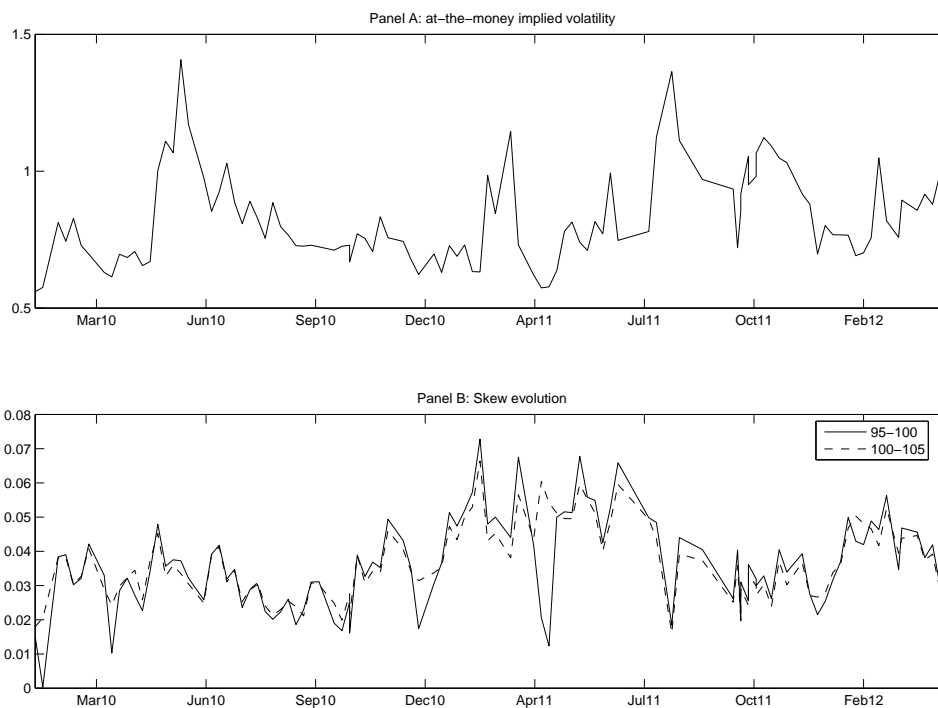
## List of tables

**Table 1:** Estimation results for the TFSV-MR model associated with the implied volatility surface of the VIX index corresponding to February 22, 2012.

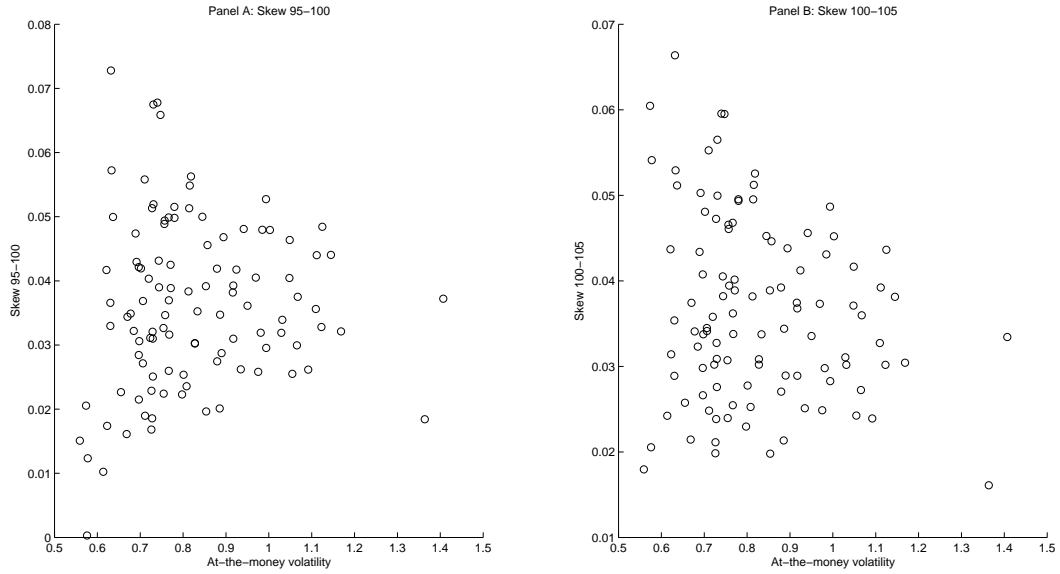
Parameter	Value	Parameter	Value
$\kappa$	2.5359	$v_1$	0.3445
$\theta$	2.8468	$\kappa_2$	11.0467
$\kappa_1$	3.8344	$\theta_2$	0.2493
$\theta_1$	0.2158	$\sigma_2$	2.9659
$\sigma_1$	3.4993	$\rho_2$	0.7138
$\rho_1$	0.9402	$v_2$	0.2718
<i>MAE implied volatilities</i>	1.4708%		
<i>Percentage MAE prices</i>	0.2915%		
<i>dollar MAE prices</i>	0.0530		

Note. The Percentage MAE, associated with option prices, is normalized by the underlying asset price.

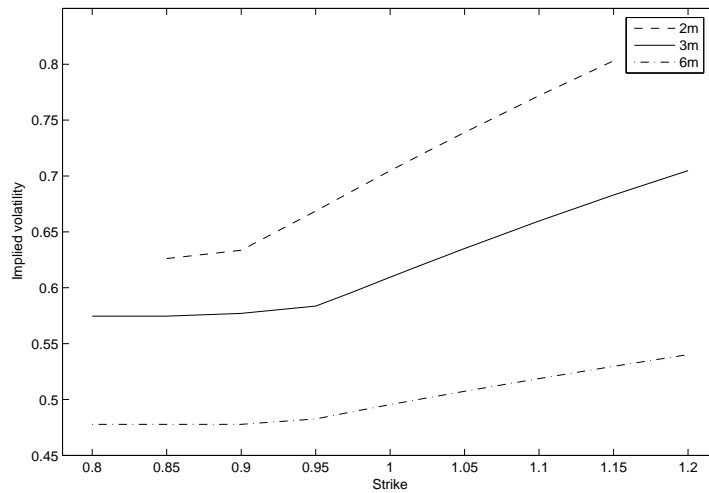
## List of figures



**Figure 1:** Weekly evolution of the at-the-money implied volatility (panel A), as well as the 95-100 skew and the 100-105 skew (panel B) associated with options with maturity within three months, during the period January 6, 2010 to April 18, 2012, for the VIX index. The 95-100 skew is defined as  $\Sigma_1 - \Sigma_{0.95}$ , whereas the 100-105 skew is defined as  $\Sigma_{1.05} - \Sigma_1$ , where  $\Sigma_K$  is the implied volatility corresponding to options with strike equal to  $K$  expressed as a percentage of the underlying asset. The data have been obtained from Bloomberg.

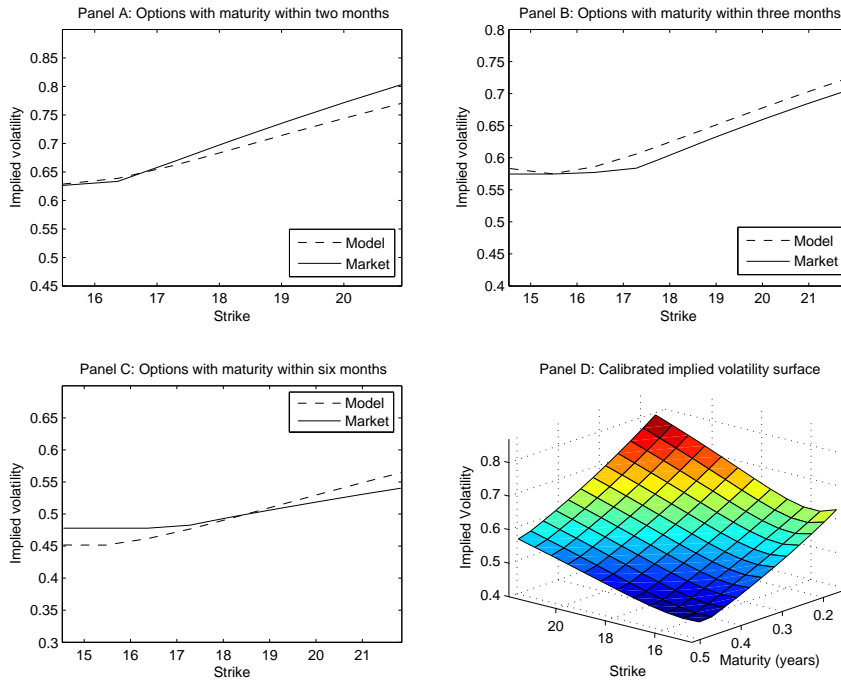


**Figure 2:** Scatter plot corresponding to the relation between the 95-100 skew and the at-the-money implied volatility (panel A), as well as to the relation between the 100-105 skew and the at-the-money implied volatility (panel B). The 95-100 skew is defined as  $\Sigma_1 - \Sigma_{0.95}$ , whereas the 100-105 skew is defined as  $\Sigma_{1.05} - \Sigma_1$ , where  $\Sigma_K$  is the implied volatility corresponding to options with strike equal to  $K$  expressed as a percentage of the underlying asset. The data have been obtained from Bloomberg.

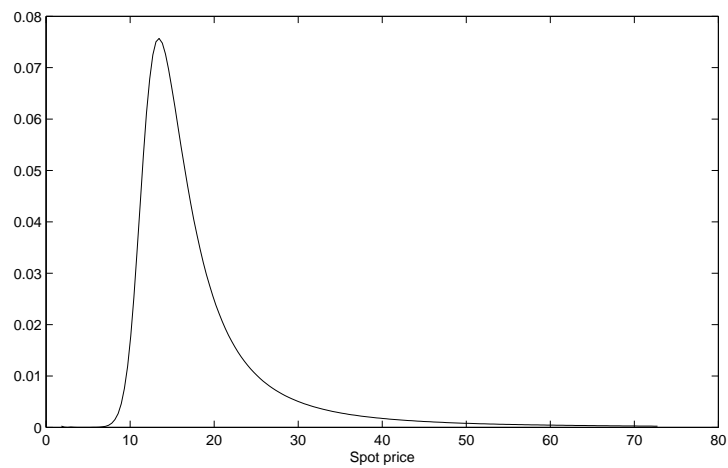


**Figure 3:** Market implied volatility skews associated with options with maturity within 2 months, 3 months and 6 months, for February 22, 2012 market data, corresponding to the VIX index. Strike prices are expressed as a percentage of the index price.

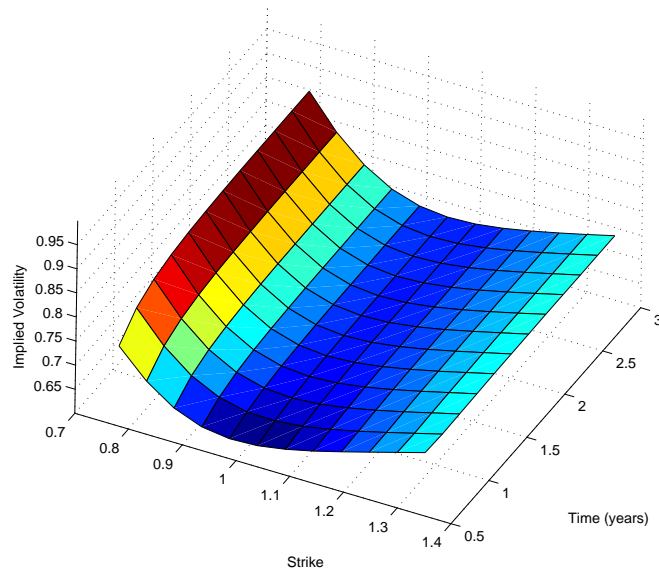




**Figure 4:** Comparison between the market implied volatility skew and the one generated by the parameters of table 1 under the TFSV-MR specification corresponding to February 22, 2012 (panel A, B and C), as well as calibrated implied volatility surface (Panel D).



**Figure 5:** Calibrated risk-neutral density function associated with options with maturity within six months for the VIX index corresponding to February 22, 2012.



**Figure 6:** Three months forward implied volatility surface generated by the TFSV-MR specification of table 1. For each strike  $K_i$ , expressed in percentage terms, and time  $t_j$ , expressed in years, the figure shows the forward implied volatilities associated with the following forward-start calls  $C_0\left(X_0, t_j - \frac{1}{4}, t_j, K_i\right)$  where  $X_0 = 18.19$ , is the spot price corresponding to the VIX index.