Extracting nonlinear signals from several economic indicators.*

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Abstract

We develop a twofold analysis of how the information provided by several economic indicators can be used in Markov-switching dynamic factor models to identify the business cycle turning points. First, we compare the performance of a fully non-linear multivariate specification (one-step approach) with the "shortcut" of using a linear factor model to obtain a coincident indicator which is then used to compute the Markov-switching probabilities (two-step approach). Second, we examine the role of increasing the number of indicators. Our results suggest that one step is generally preferred to two steps, although its marginal gains diminish as the quality of the indicators increases and as more indicators are used to identify the non-linear signal. Using the four constituent series of the Stock-Watson coincident index, we illustrate these results for US data.

Keywords: Business Cycles, Output Growth, Time Series.

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1 Introduction

The view in the two decades leading up to 2007 was that macroeconomic policy had advanced to the point of guaranteeing smooth business cycles, considerably decreasing the probability of tail risks associated with sharp reductions in output and employment. However, this buoyant view was called into question when a financial crisis erupted during the second half of 2007, leading to one of the sharpest, longest and most generalized downturns since the Great Depression. With some lag after the trough, governments and central banks embarked on aggressive fiscal and monetary policies to limit in advance the fall of economic activity. Soon after the peak, the economic authorities were progressively abandoning their economic stimulus packages as the adverse symptoms of the recession abated. Since the decisions about the size and timing of these policies were conducted in real time, the 2008-2009 recession has been the source of a revival of econometric methods that help policymakers in the early tracking of economic developments.

With the aim of producing readily interpretable signals about the ongoing economic evolution, some of these methods were based on statistical algorithms that try to capture the course of the two business cycle features embedded in the seminal description developed by Burns and Mitchell (1946) and observed in the dynamics of the economic indicators during the Great Recession. The first feature of the business cycle is the comovement among individual economic indicators. The models proposed to capture this feature usually follow the lines initiated by Stock and Watson (SW, 1991), who proposed a single-index linear dynamic factor model to analyze the comovements among industrial production, employment, income and sales. These four series have a common element that can be modeled by an underlying unobserved variable representing the overall economic activity, as in the Composite Index of Coincident Economic Indicators (CEI) of the Bureau of Economic Analysis (currently published by the Conference Board). Recent extensions of their dynamic factor model are the Aruoba, Diebold and Scotti (ADS, 2009) index of business conditions and the Chicago Fed National Activity Index (CFNAI), which also provide accurate signals about the current state of the business cycle.[1]

[1] The Euro-STING model of Camacho and Perez Quiros (2010) is the European extension of these models.

The second feature of the business cycle is the existence of two separate business cycle phases, and the models proposed to capture this notion frequently follow the lines suggested by Hamilton (1989). This author proposed a statistical method that can be applied to economic indicators whose dynamics evolve according to the outcome of a two-state Markov process with occasionally discrete shifts. The model has the advantage of automating the dating of business cycle turning points by inferring recession probabilities from the evolution of the economic indicators. Recent extensions of Markov-switching techniques used to infer the probability that economy is in recession can be found in the survey by Hamilton (2011).

In this context, two alternative approaches have been used in the literature to unify the notions of comovements and business cycle asymmetries. The description of the first approach dates back to the mid-nineties in the seminal proposal of Diebold and Rudebusch (1996) and consists of a *two step* estimation procedure. The first step is based on computing a coincident indicator, such as SW, CEI, ADS, or CFNAI, by applying linear factor models to a set of coincident indicators. In the second step, univariate Markov-switching techniques are applied to the coincident indicator to infer the underlying business cycle probabilities. Examples of recent applications of Markov-switching techniques to linear factors are Diebold and Rudebusch (1996) in the case of SW; Brave and Butters (2010) in the case of a high frequency index such as ADS; Davig (2008) in the case of CFNAI; and Paap, Segers, and van Dijk (2009) in the case of CEI.

The second approach, which was initially proposed by Kim and Yoo (1995), Chauvet (1998) and Kim and Nelson (1998), is based on the natural extension of full dynamic-factor/Markov-switching models which is estimated in *one step*. In their Markov-Switching Dynamic Factor Model (MS-DFM), comovements and business cycles are modeled with a nonlinear dynamic factor model whose common component is governed by an unobservable regime-switching variable which controls the business cycle dynamics. Recently, Chauvet and Hamilton (2006) and Chauvet and Piger (2008) examined the empirical reliability of these models in computing real-time inferences of the US business cycle.

Although the unified representation of the MS-DFM estimated through the one-step approach to infer recession probabilities is conceptually appealing, it is very tempting to fit a linear DFM to the economic indicators and a univariate Markov-switching model to the resulting linear coincident indicator. One reason is that the linear coincident indicators, such as ADS, CFNAI and CEI, are already constructed by several agencies and it seems straightforward to use univariate Markov-switching filters to compute state probabilities from them. Another reason is that the numerical algorithms used to evaluate the likelihood functions of the (one-step) MS-DFM usually suffer from curse of dimensionality problems, which may hamper its empirical implementation, especially in real time.

In spite of the coexistence of the two approaches, the analysis of the performance of the one-step procedure with respect to the two-step approach has not still been addressed in the literature. To provide some light to help us to fill in this gap, we examine the sources of misspecification of applying Markov-switching models to the common factor of a linear DFM when the data generating process is a nonlinear MS-DFM. Our conjecture is that the two-step procedure faces greater difficulties to infer business cycle probabilities when the Kalman filter used to compute the linear factor model in the first step assigns large weights to the past observations of the economic indicators used in the business cycle analysis. Using the Riccati equation for the misspecified linear Kalman filter, we show that this typically occurs when the quality of the indicators used in the analysis is relatively limited, which basically occurs when the indicators are noisy. However, when the economic indicators are carefully selected to have large signal-to-noise ratios in the Kalman filter, the empirical performance of the one-step procedure is not expected to be highly superior to that of the two-step method. These theoretical results are confirmed by means of a Monte Carlo experiment.

In addition, we examine the extent to which inferences about the state of the economy from nonlinear MS-DFM can be improved upon by including additional variables. In a linear framework, the recent literature provides mixed evidence about how many series to consider for forecasting. Although consistency results are available when using principal components as estimators of the common factors, as both N (the number of series) and T (the sample size) tend to infinity, Boivin and Ng (2006) were among the first to show that the empirical forecasting performance of these models does not necessarily improve upon with N. In the nonlinear context of this paper, we show that the precision of the inference about the business cycle is expected to grow by including additional indicators, with larger growth for less noisy indicators. However, regarding the quality of the new indicators, the expected gains from the additional indicators are progressively lower as the number of indicators already included becomes large. In empirical applications, this result implies that the gains from using large sets of indicators can be deceptively low compared to the computational complexity of dealing with nonlinear models that use many indicators.

Finally, we evaluate the empirical relevance of our results by evaluating the performance of the one step and two step estimation procedures to infer the US business cycles from the four coincident economic indicators highlighted by Stock and Watson (1991) considered together and individually. Our main results are the following. First, we show that although the individual component indicators track the business cycle fairly well, their performance is inferior to that of the two versions of the multivariate coincident indicators. Second, as expected, since the indicators exhibit very high signal-to-noise ratios, the dynamics of the coincident indicators estimated from the two estimation procedures are both in close agreement. Third, the filtered probabilities estimated from the two estimation procedures are also in striking accord with the NBER business cycle chronology. Meanwhile, the two methods exhibit similar mean square error measures when comparing the recession probabilities with a dummy that takes the value of one in the NBER recessions. Interestingly, when the analysis is restricted to the first month of the new state after a business cycle phase shift, the relative errors achieved by the full Markov-switching dynamic-factor model (the one step model) imply considerable reductions of mean square errors of about 20%. Notably, the reductions stand at about 30% when the comparative analysis focuses on the first month of each new expansion. This result is confirmed in a real-time exercise, where the recursively increasing databases used in each month incorporate only data that would have been available in the month being considered. In this case, the relative gains of the one-step estimation procedure when interpreting the signals of business cycle phase shifts in the first months after the troughs are even larger.

The structure of this paper is organized as follows. Section 2 describes the sources of misspecification of a two-step estimation procedure when the data generating process is a nonlinear MS-DFM. In addition, it examines the extent to which the model performance can be improved upon by using additional indicators. Section 3 proposes a Monte Carlo experiment to analyze the ability of the models in computing business cycle inferences. Section 4 shows the empirical analysis of US business cycles. Section 5 concludes.

2 Two-step versus one-step approaches

This section examines the performance of two-step versus one-step estimation procedures of dynamic factor models with Markov-switching to accurately detect the probability of a given business cycle phase. For this purpose, the section investigates the sources of misspecification of the two-step estimation method and examines the extent to which the accuracy to infer the business cycle state can be improved upon by enlarging the number of indicators.

2.1 Sources of misspecification

This section examines the sources of misspecification of the two-step estimation method when the procedure is used to infer the probability of a given business cycle phase. For this purpose, let us assume that the data generating process is a nonlinear MS-DFM but that an analyst erroneously fits a linear single-index DFM and tries to infer the recession probabilities from the resulting common factor with univariate Markov-switching techniques.

Let $\mathbf{y}_t = (y_{1,t}, ..., y_{N,t})'$ be the vector of N economic indicators which admits a factor decomposition into a non-observed common factor f_t and N specific or idiosyncratic components:

$$\underset{N\times 1}{\mathbf{y}_t} = \underset{N\times 1}{\mathbf{\Lambda}} \underset{1\times 1}{f_t} + \underset{N\times 1}{\mathbf{u}_t}, \quad (1)$$

where $\mathbf{\Lambda} = (\lambda_1, \lambda_2, ..., \lambda_N)'$ is the vector of factor loadings. Let us assume that $\mathbf{u}_t$ is a multivariate Gaussian white noise with mean equal to $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}_u$. As in classical factor analysis, $\mathbf{\Sigma}_u$ is assumed to be a diagonal matrix with the vector of variances $(\sigma_1^2, \sigma_2^2, ..., \sigma_N^2)$ in its main diagonal.

To complete the specification of the data generating process, the factor is assumed to be governed by an unobserved regime-switching mean plus a noise

$$f_t = \mu_{s_t} + a_t, \tag{2}$$

where a_t is univariate Gaussian white noise $(0, \sigma_a^2)$.[2] Within this framework, one can label $s_t = 0$ as expansions and $s_t = 1$ as recessions at time t if $\mu_0 > \mu_1$. In these cases, the common dynamics of the coincident economic indicators are expected to exhibit high (usually positive) growth rates in expansions and low (usually negative) growth rates in recessions. In addition, s_t is assumed to evolve according to an irreducible 2-state Markov chain whose transition probabilities are defined by

$$p(s_t = j | s_{t-1} = i, s_{t-2} = h, ..., I_{t-1}) = p(s_t = j | s_{t-1} = i) = p_{ij}, \tag{3}$$

where $i, j = 0, 1$, and I_t is the information set up to period t.

Instead of fitting the one-step MS-DFM described above, let us assume that the analyst erroneously applies a two-step MS-DFM as follows. In the first step, the analyst estimates a linear DFM to the set of N economic indicators whose common factor is assumed to follow a simple autoregressive process of order one to facilitate the analysis. Accordingly, the analyst computes a misspecified common factor, which is denoted with an asterisk,

$$f_t^* = d + \phi f_{t-1}^* + a_t^*. \tag{4}$$

In this expression, a_t^* is a univariate white noise with zero mean and variance $\sigma_{a^*}^2 = 1$, which agrees with the standard identification assumption. The intercept d is added to take into account the possibility of a non-zero unconditional mean. The autoregressive parameter, ϕ, captures the serial correlation induced by the switching mean of the common factor described in (2).

Once $f_{t|t}^*$ is estimated in the first step, the analyst applies a univariate Markov-switching model to the common factor in the second step. Hence, the analyst estimates the nonlinear model by approximate maximum likelihood techniques and obtains the filtered

[2] Qualitatively similar results were obtained from more complex dynamics of the common factor. In particular, we performed simulations by assuming autoregressive processes for the series a_t, as well as for the idiosyncratic noises $u_{i,t}$, $i = 1, ..., N$. Some of the Monte Carlo results are reported in Section 3.

state probabilities which are computed to extract informative insights about the business cycle. Following Hamilton (1989), these probabilities can be expressed as

$$prob(s_t = j|I_t^*) = \frac{f(f_{t|t}^*|s_t = j, I_t^*)prob(s_t = j|I_{t-1}^*)}{f(f_{t|t}^*|I_t^*)}, \tag{5}$$

where $f(\bullet)$ is the Gaussian density function and $I_t^* = \left\{(f_{\tau|\tau}^*)_{\tau=1}^{\tau=t}\right\}$. Therefore, the two-step estimation procedure faces a misspecification problem which might be of potential considerable importance in the detection of business cycle turning points.[3] Since $f_{t|t}^*$ is a linear combination of past and present values of $\mathbf{y}_t$, the density functions used in the univariate Markov-switching model depend not only on the current state but on all past and present states. Accordingly, the simple two-step method might underweight the signals of imminent changes in business cycle phases, implying longer delays in signaling a new business cycle phase.

To understand this statement, recall that the filtered linear common factor can be expressed as a weighted sum of past and present observations

$$f_{t|t}^* = \sum_{\tau=1}^{t} \mathbf{w}_{t,\tau}\mathbf{y}_\tau, \tag{6}$$

where the weights $\mathbf{w}_{t,\tau}$ are the N-dimensional row vectors given by:

$$\begin{aligned} \mathbf{w}_{t,t} &= \frac{1}{c_t}\mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}, \\ \mathbf{w}_{t,\tau} &= \frac{1}{c_\tau}\frac{1}{V_{\tau|\tau-1}}\phi\mathbf{w}_{t,\tau+1} = B_\tau\phi\mathbf{w}_{t,\tau+1}, \end{aligned} \tag{7}$$

for $\tau = t-1, ..., 1$. In the last expression, $V_{\tau|\tau-1}$ is the mean squared error of the misspecified state estimated at τ with information up to $\tau - 1$, $c_\tau = \frac{1}{V_{\tau|\tau-1}} + \mathbf{\Lambda}'\mathbf{\Sigma}_u^{-1}\mathbf{\Lambda}$, and $B_\tau = \frac{1}{c_\tau}\frac{1}{V_{\tau|\tau-1}}$.

Since the weights are decreasing in the signal extraction of an autoregressive stationary process through the Kalman filter, the misspecification problem of the one-step estimation procedure is expected to be high when the weights decay relatively slow. In these cases, the approximation of the univariate Markov-switching model applied to the linear common

[3] To facilitate the analysis, we assume known population parameters and that the only source of misspecification comes from the way the common factor is extracted. Our simulations confirm that this is a very reasonable assumption.

factor will lead to important reductions in the timely detection of current turning points since the information of a potential regime switch contained in $\mathbf{y}_t$ diminishes its impact on $f^*_{t|t}$. Hence, a good strategy to examine the sources of misspecification of the two-step estimation procedure is to analyze the sources of persistence of the linear common factor which, according to expression (7), depends on B_τ and ϕ.

Focusing on B_τ, the larger it is, the stronger the misspecification of the two-step procedure becomes. To find the coefficients governing B_τ, it is worth noting that the linear filter reaches its steady state and $V_{t|t-1} = V$ since $\phi < 1$. Hence, the solution of the algebraic Riccati equation for the misspecified filter is

$$V = \frac{\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2} - (1-\phi^2) + \sqrt{\left(\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2} - (1-\phi^2)\right)^2 + 4\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2}}}{2\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2}}, \tag{8}$$

which implies that B_τ is

$$B_\tau = \frac{2}{\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2} + (1+\phi^2) + \sqrt{\left(\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2} - (1-\phi^2)\right)^2 + 4\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2})}}. \tag{9}$$

This expression reveals that the greater the autoregressive parameter ϕ and the sum $\sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2}$, the smaller B_τ. To interpret the ratios $\frac{\lambda_i^2}{\sigma_i^2}$, it is worth writing the common factor as

$$f^*_{t|t} = \frac{1}{c_t}\left(\frac{1}{V_{t|t-1}} f^*_{t|t-1} + \sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2}(\frac{y_{i,t}}{\lambda_i})\right), \tag{10}$$

with $c_t = \frac{1}{V_{t|t-1}} + \sum_{i=1}^N \frac{\lambda_i^2}{\sigma_i^2}$. This expression states that $f^*_{t|t}$ is the weighted sum of two components. The first component, $f^*_{t|t-1}$, is the estimation of the factor at time t with the information up to time $t-1$, and it has a weight which is proportional to the precision of this estimation. The second component is the weighted sum of the new information incorporated by the indicators observed at t, $\frac{y_{i,t}}{\lambda_i}$.[4] The weights , $\frac{\lambda_i^2}{\sigma_i^2}$, $i = 1, ..., N$, are called signal-to-noise ratios since they measure the precision of the indicators (inverse of the conditional variances $var(f_t|y_{i,t})$, $i = 1, ..., N$). Hence, the signal-to-noise ratios assign more weight to compute $f^*_{t|t}$ to the less noisy economic indicators. In addition, when the signal-to-noise ratios are low, the weights to the past information, $\mathbf{w}_{t,\tau}$, become

[4] Notice that $\frac{y_{i,t}}{\lambda_i}$ is the conditional expectation of the common factor from the i-th indicator $E(f_t|y_{i,t})$.

large. Summing up, using economic indicators with low signal-to-noise ratio leads the approximation of univariate Markov-switching dynamics of the common factor used in the linear DFM estimation of the common factor to become increasingly more inappropriate to detect business cycle turning points in advance.

The influence of ϕ on the weights is twofold. According to expression (7), this parameter increases past weights directly. At the same time, it reduces the past weights through B_τ. Hence, the net effect of ϕ on business cycle identification is complex and we refer the readers to the simulation experiment developed in Section 3 for a more detailed analysis. In spite of this unclear effect, Timmerman (2000) showed that

$$\phi = \frac{(\mu_0 - \mu_1)^2 \pi_1 \pi_0 (p_{00} + p_{11} - 1)}{(\mu_0 - \mu_1)^2 \pi_1 \pi_0 + \sigma_a^2}, \tag{11}$$

where π_i is the steady state probability of state i, such that $\pi_1 + \pi_0 = 1$, and

$$\pi_i = \frac{1 - p_{jj}}{2 - p_{ii} - p_{jj}}, \tag{12}$$

with $i, j = 0, 1$. Accordingly, the autoregressive parameter is an increasing function on the difference of the within-state means and on the persistence of the business cycle states.[5] Assuming that $\mu_0 > \mu_1$, this means that the larger the difference between the two conditional means (that is, if $\mu_0 >> \mu_1$), the larger ϕ should be and this will help to identify the business cycle regimes since it separates the Gaussians of the mixture (see Chauvet and Hamilton, 2006).

The effects of relaxing the assumptions about the dynamics of the idiosyncratic components, $\mathbf{u}_t$, deserve a final comment. In expression (1), we assumed that the idiosyncratic components followed a multivariate white noise. However, this assumption can be relaxed by appropriately defining $\mathbf{\Sigma}_u$ in the previous expressions. For instance, let us assume that the idiosyncratic components follow the diagonal VAR(1) process

$$\mathbf{u}_t = \Psi \mathbf{u}_t + \boldsymbol{\epsilon}_t, \tag{13}$$

where $\mathrm{var}(\boldsymbol{\epsilon}_t) = diag(\sigma_1^2, ..., \sigma_N^2)$, and $\Psi = diag(\psi_1, ..., \psi_N)$. In this case, $\mathbf{\Sigma}_u = diag(\frac{\sigma_1^2}{1-\psi_1^2}, ..., \frac{\sigma_N^2}{1-\psi_N^2})$. Notice that the conditional densities of the observed series will depend on the hidden state

[5] It can be easily checked that for given probabilities, $p_{ii}, i = 0, 1$, the derivative of ϕ with respect to $(\mu_0 - \mu_1)^2$ is always positive.

contemporaneously and through its first lags. We check how this affects our results on the simulations presented on Section 3.

2.2 The role of N

Empirical applications of MS-DFM frequently exhibit the typical curse of dimensionality problems of nonlinear estimates. This precludes the analysts from considering the case of large values of N. In spite of this comment, the question of how many economic indicators are useful to compute accurate inferences of business cycle turning points still holds. If an analyst starts with a set of $N-1$ economic indicators that provides reasonable turning point signals, the problem reduces to the question of under which circumstances the additional N-th variable may be incorporated into the model leaving the dimension of the resulting nonlinear model manageable.

For this purpose, we consider that the set of N indicators is preferred to the set of $N-1$ indicators if the former sufficiently increases the ability to appropriately detect true turning points and reduces the rate of false signals. Let us denote the set of $N-1$ indicators by $I_{1,t} = \{y_{1,1}, ..., y_{1,t}; y_{2,1}, ..., y_{2,t}; ...; y_{N-1,1}, ..., y_{N-1,t}\}$, and the N-th indicator by $I_{2,t} = \{y_{N,1}, ..., y_{N,t}\}$. Hence, we find it useful to include the last indicator in inferring the state of the economy at time t whenever $prob(s_t = 1|I_{1,t}, I_{2,t}) > prob(s_t = 1|I_{1,t})$ when $s_t = 1$ (for instance, recessions) and $prob(s_t = 1|I_{1,t}, I_{2,t}) < prob(s_t = 1|I_{1,t})$ when $s_t = 0$ (for instance, expansions). Since it is straightforward to show that if $prob(s_t = i|I_{1,t}) = 1$, then $prob(s_t = i|I_{1,t}) = prob(s_t = i|I_{1,t}, I_{2,t}) = 1$, let us assume that $0 < prob(s_t = i|I_{1,t}) < 1$.

To start with, let us consider that the quality of the N-th indicator is similar to that of the set that contains the $N-1$ first indicators, i.e., that it is not a noisier time series. Let us focus the analysis on the identification of a given regime, for example on $s_t = 1$.[6] According to the Markov-chain properties of the model described in (1) to (3), $prob(s_t = 1|I_{1,t}, I_{2,t}) = prob(s_t = 1|y_{1,t}, ..., y_{N,t})$ and $prob(s_t = 1|I_{1,t}) = prob(s_t = 1|y_{1,t}, ..., y_{N-1,t})$.

[6] The treatment of regime $s_t = 0$ is symmetric.

Then, the inference computed from the set of N indicators can be expressed as

$$\begin{aligned} prob(s_t &= 1|I_{1,t}, I_{2,t}) = \frac{prob(s_t = 1, y_{N,t}|I_{1,t})}{f(y_{N,t}|I_{1,t})} \\ &= \frac{f(y_{N,t}|s_t = 1, I_{1,t})}{f(y_{N,t}|I_{1,t})} prob(s_t = 1|I_{1,t}) \\ &= w_t prob(s_t = 1|I_{1,t}), \end{aligned} \tag{14}$$

where $w_t = \frac{f(y_{N,t}|s_t=1,I_{1,t})}{f(y_{N,t}|I_{1,t})}$. Therefore, the information of the last indicator $y_{N,t}$ will be useful to compute inferences about the business cycle at time t if the weight $w_t > 1$ given that the true regime is $s_t = 1$.

It is useful to express the weights as

$$w_t = \frac{f(y_{N,t}|s_t = 1, I_{1,t})}{f(y_{N,t}|s_t = 1, I_{1,t}) \times prob(s_t = 1|I_{1,t}) + f(y_{N,t}|s_t = 0, I_{1,t})(1 - prob(s_t = 1|I_{1,t}))}. \tag{15}$$

Then, there exists informational content in the N-th indicator, i.e, $w_t > 1$ when $s_t = 1$, if

$$f(y_{N,t}|s_t = 1, I_{1,t}) > f(y_{N,t}|s_t = 1, I_{1,t}) \times prob(s_t = 1|I_{1,t}) + f(y_{N,t}|s_t = 0, I_{1,t})(1 - prob(s_t = 1|I_{1,t})). \tag{16}$$

This occurs whenever

$$\frac{f(y_{N,t}|s_t = 1, I_{1,t})}{f(y_{N,t}|s_t = 0, I_{1,t})} > 1. \tag{17}$$

Since the inequality in (17) does not necessarily hold for all possible values of $y_{N,t}$ (for instance, in the case of overlapping density functions) it will be useful to evaluate if this condition holds on average. Taking natural logarithms of this expression, the set of N indicators outperforms on average the set of $N-1$ indicators if

$$\ln f(y_{N,t}|s_t = 1, I_{1,t}) - \ln f(y_{N,t}|s_t = 0, I_{1,t}) > 0 \tag{18}$$

when $s_t = 1$. Taking into account all possible outcomes of $y_{N,t}$ when $s_t = 1$, the expected value of the difference between the two conditional densities under conditional Gaussianity is given by

$$\begin{aligned} \int \ln f(y_{N,t}|s_t &= 1, I_{1,t}) f(y_{N,t}|s_t = 1, I_{1,t}) dy_{N,t} - \\ - \int \ln f(y_{N,t}|s_t &= 1, I_{1,t}) f(y_{N,t}|s_t = 1, I_{1,t}) dy_{N,t} > 0. \end{aligned} \tag{19}$$

Therefore, one could evaluate if condition (17) is fulfilled on average.

The next proposition, whose proof appears in the Appendix, quantifies the expected gains in terms of business cycle identification of adding a new economic indicator $y_{N,t}$ when $s_t = 1$ to a given set of $N-1$ indicators. The magnitude of the change is a measure of the averaged informational content of $y_{N,t}$ about the two states (its ability to separate them) and uses the concept of conditional entropy (or Kullback-Leibler divergence).

Proposition 1 *The Kullback-Leibler (KL) divergence of $f(y_{N,t}|s_t = 0, I_{1,t})$ with respect to $f(y_{N,t}|s_t = 1, I_{1,t})$ under the MS-DFM assumptions described in (1) to (3), is given by*

$$KL = \frac{\lambda_N^2(\mu_0 - \mu_1)^2}{2\sigma_N^2} \frac{\frac{1}{\sigma_a^2}}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1} \frac{\lambda_i^2}{\sigma_i^2}} \frac{\frac{1}{\sigma_a^2}}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N} \frac{\lambda_i^2}{\sigma_i^2}}. \tag{20}$$

This expression implies that (i) if there are separate business cycles regimes, in the sense that $\mu_1 \neq \mu_0$ and $\sigma_a^2 < \infty$; and (ii) if the new indicator is informative, in the sense that $\lambda_N \neq 0$ and $\sigma_N^2 < \infty$, then the divergence is strictly positive. This implies that adding a new indicator is (on average) always useful in terms of business cycle identification.

In spite of this result and due to the curse of dimensionality problem of nonlinear models, it is interesting to quantify the informational content of the new indicator which is measured by the magnitude of the KL divergence. Two interesting results deserve special comments from this proposition. First, the informational content of the additional N-th indicator increases with the signal-to-noise ratio of this indicator $\frac{\lambda_N^2}{\sigma_N^2}$. Hence, the accuracy of the model to provide clear business cycle signals increases with the quality of the new indicator. Second, assuming that the signal-to-noise ratio is the same for all the economic indicators, the divergence is a decreasing function of the number of indicators.

According to these two comments, it is worth emphasizing that the gains of adding new indicators can be lower than proportional to the number of indicators which have already been included to infer the business cycle probabilities. Hence, the decreasing ability to improve upon the accuracy of the model cannot always be compensated with the quality of the new indicator. This implies that the gains in terms of divergence from using large sets of indicators can be deceptively low compared with the computational complexity of handling with many indicators in empirical applications of nonlinear models.

Figure 1 helps us to interpret the size of the gains as a function of both the signal-to-noise ratio of the new indicator and the number of indicators already included in the model. To facilitate comparisons we set $\mu_0 - \mu_1 = \sqrt{2}$ and $\sigma_a^2 = 1$, and we assume that the signal-to-noise indicators included in $I_{1,t}$ is 1 for all of its $N-1$ indicators. Using these assumptions, the figure plots the KL divergence of $f(y_{N,t}|s_t = 0, I_{1,t})$ with respect to $f(y_{N,t}|s_t = 1, I_{1,t})$ when the additional indicator exhibits potential signal-to-noise ratios from 0.1 to 40. Given a number of indicators, for instance $N = 2$, the figure shows that the divergence increases rapidly when the new indicator exhibits signal-to-noise ratios of up to about six times the signal-to-noise ratios of the existing indicators. However, the divergence becomes hump-shaped at these values, implying that using less noisy indicators does not help to increase the divergence by a large amount.

To evaluate the role of the number of indicators, the figure also plots the results from repeating the previous exercise when the final number of indicators is also $N = 5$ and $N = 8$. The figure shows that the divergence functions are shifted down by increasing N. This implies that although enlarging the original set of indicators with a new indicator of fixed signal-to-noise ratio leads to improve the accuracy to infer the business cycle phases, the gains are low when the initial set of indicators becomes large.

3 Monte Carlo simulations

In this section, we set up several Monte Carlo experiments to study how the data might affect the empirical performance of one-step versus two-step estimation procedures as well as the role of N to infer business cycle probabilities. For this purpose, we generate a total of $M = 1000$ sets of N idiosyncratic components $\mathbf{u}_t^m$ of length $T = 200$ and equal variances $\sigma_i^2 = \sigma^2$. The dynamics of these time series are assumed to follow autoregressive processes of order one with autoregressive parameters equal to 0.3.

Besides, we generate $M = 1000$ dummy variables b_t^m of zeroes and ones of length $T = 100$ which are used to simulate different sequences of expansions and recessions. To ensure that the dummies share the US business cycle properties, we assume that b_t^m follows Markov chains with $p_{00} = 0.9$ and $p_{11} = 0.7$, which are the percentage of quarters classified as expansions that are followed by expansions and the percentage of quarters classified as recessions that are followed by recessions in the period 1959.3-2010.3 by the NBER, respectively. Hence, we generate $M = 1000$ common factors that follow Markov-switching processes f_t^m by using the business cycle sequences b_t^m and by assuming that $\sigma_a^2 = 1$.

Using factor loadings equal to one for all the series, we add the idiosyncratic components to the switching mean factors to generate $M = 1000$ sets of time series $\mathbf{y}_t^m$. Then, we apply both the two-step and one-step estimation procedures to extract the filtered probabilities of state 1, $p_{t,i}^m$, with $i = I, II$ in the cases of using one-step and two-step estimation procedures, respectively. The Monte Carlo experiment is developed for $N = 3$ indicators of different quality. In particular, we generate indicators of different differences of the within-state means ($\mu_0 - \mu_1 = 1, 2, 4$ and 10), and different variances ($\sigma^2 = 0.5, 1.5$ and 4.5).

For each m-th replica, we quantify the ability of these two estimation procedures to detect the actual state of the business by computing the Quadratic Probability Score (QPS, from now on):

$$QPS_i = \frac{1}{M}\sum_{m=1}^{M}\frac{1}{T}\sum_{t=1}^{T}(p_{ti}^m - b_t^m)^2, \tag{21}$$

where $i = I$ in the case of the one-step estimation procedure, and $i = II$ in the case of the two-step estimation procedure. This measure can be interpreted as the average over the M replications of the squared deviation from the generated business cycles.

To examine the sources of misspecification of the two-step estimation method when the procedure is used to infer the business cycle probabilities instead of the one-step procedure, Table 1 displays the QPS statistics which are computed for the different scenarios described above. To examine the ability of each model to detect turning points, the table shows in parentheses the scores when QPS is calculated only for those t that refer to the first period after the phase shifts.[7]

The main message of this table is that the one-step estimation procedure unequivocally performs better than the two-step estimation procedure. However, the relative per-

[7] Recall that turning point detection is a key business cycle question in the forecasting arena. The empirical section also devotes special attention to this topic.

formance gains depend on the quality of the indicators used in the business cycle analysis. For a given idiosyncratic variance, higher differences of within-state means (from 1 to 10) improve the performance of both one-step and two-step estimation procedures. In spite of this result, it is worth pointing out that when the differences of within-state means become large enough, although the one-step approach improves proportionally more than the two-step procedure, the gains are statistically important but not economically meaningful since the misspecified two-step approach is already very accurate. For example, for a variance of $\sigma^2 = 1.5$, the ratio of the QPS statistics between the two-step and the one-step procedures when $\mu_0 - \mu_1 = 1$ is 1.21 ($QPS_{II} = 0.246$ versus $QPS_I = 0.202$) while the ratio is more than 4000 when $\mu_0 - \mu_1 = 10$ ($QPS_{II} = 0.019$ versus $QPS_I = 4.22E$-06). However, the QPS of the two-step approach was very low already (0.019).[8] The results also hold for the turning point detection whose results are displayed in parentheses in Table 1. In this case, the table also shows that when the set of indicators included in the analysis are very precise in terms of signal-to-noise ratios, there is less room for the one-step method to improve the empirical performance of the two-step procedure. On the contrary, if the indicators are not so good, the empirical performance of the one-step procedure can significantly outperform that of the two-step procedure.

>From Table 2, which examines the role of N in the one-step estimation procedure performance, there are two noteworthy findings that deserve comments. First, the table shows that the number of indicators used to infer the business cycle phases matters since increasing the number of time series leads to business cycle identification improvements, i.e,., QPS reductions. However, the usefulness of new indicators in terms of business cycle identification refinements dramatically decreases when the number of indicators already used becomes large. For example, for $\sigma^2 = 1.5$, a model of $N = 1$ indicator exhibits $QPS_I^{N=1} = 0.168$, and adding two more indicators implies an improvement of 26% ($QPS_I^{N=3} = 0.124$). However, the improvement falls to 12% when the set of indicators is extended from $N = 3$ to $N = 5$, and it is only 9% when the number of indicators is enlarged from $N = 5$ to $N = 7$. Second, the table highlights that the quality of the

[8] To make the QPS results more readable, recall that the inference that gives probability of recession equals to zero for all t leads to a QPS of 0.20.

indicators matters when one is interested in quantifying the expected gain from enlarging the model. In that sense, when the indicators are very precise (for instance, $\sigma^2 = 0.5$), the gain of enlarging the model from $N = 3$ to $N = 7$ indicators is only 9%. However, when the indicators are not very precise (for instance, $\sigma^2 = 4.5$), using $N = 7$ instead of $N = 3$ indicators increases the expected accuracy by more than 27%. The intuition for this result is that the model with $N = 3$ noiseless indicators is able to infer the business cycle with considerable accuracy. Hence, a model that uses these three indicators does not leave too much room for any improvement when the number of indicators is enlarged. In the case of computing business cycle inferences from noisier indicators, there are larger potential accuracy gains from a model with increased dimension.

4 Empirical results

The purpose of this section is to examine the empirical performance of the one-step versus the two-step estimation procedures and the role of combining information from a set of economic indicators by using an updated real-time version of the dataset previously used by Stock and Watson (1991), Chauvet (1998) and Chauvet and Piger (2008). The four indicators used in the empirical analysis, whose logarithms are plotted in Figure 2, are monthly industrial production index (IP), nonfarm payroll employment (EMPL), personal income less transfer payments (INC) and real manufacturing and trade sales (SALES) from 1967.01 to 2010.11.

Clearly, the behavior of these series clearly shows the comovements that were designated by Burns and Mitchell (1946) as the first business cycle feature. Based on the notion that these comovements have a common element that can be captured by a single underlying unobserved variable, Stock and Watson (1991) adopted a single-index linear dynamic factor model to estimate these particular dynamics. Along these lines, we fit a factor model to one hundred times the change in the natural logarithm of these four macroeconomic variables.[9]

Interestingly, the maximum likelihood estimates, which are displayed in Table 3, show

[9] According to Stock and Watson (1991), all the linear autoregressive processes are estimated with two lags. According to Camacho and Perez Quiros (2007), the nonlinear factor is estimated with no lags.

that the signal-to-noise ratios, which we defined as λ^2/σ^2, are in line with the values used in our simulations. Recalling that the simulations were made with $\lambda = 1$, the signal-to-noise ratio of IP, EMPL, INC and SALES are $1.83, 1.04, 0.09$, and 0.34, which would correspond in the simulations to the cases of $\sigma^2 = 0.5$ (IP), $\sigma^2 = 1.5$ (EMPL) and $\sigma^2 = 4.5$ (INC and SALES). The fact that the magnitude of the parameters chosen in the simulations exercise matches the data reinforces the learnings of the simulations exercise whose results can directly be applicable to the empirical analysis. In addition, the estimates show that the factor loadings are positive and statistically significant. Hence, the indicators are positively correlated with the estimated common factor, which in Burns-Mitchell's terminology represents the reference cycle. In line with this statement, Figure 3 shows that the coincident index describes a behavior that closely agrees with the NBER-designated US business cycles.[10]

Figure 2 can also be used to illustrate the second attribute of the business cycle described by Burns and Mitchell (1946): the distinction of two separate business cycle phases. Clearly, the variables depicted in this figure present an upward trend. However, this trend does not seem to be a smooth curve but rather a sequence of upturns and downturns that are closely related to the NBER business cycles phases. In this respect, although the overall average monthly growth rates of production, employment, income and sales are positive (0.19, 0.13, 0.25, 0.21, respectively), they are negative during the NBER recessions (-0.70, -0.18, -0.06 and -0.56, respectively) which correspond to periods of negative growth.

The common factor depicted in Figure 3, which is a linear combination of the four key indicators, exhibits a remarkable business cycle pattern. Accordingly, Diebold and Rudebusch (1996) suggested that the Markov-switching model proposed by Hamilton (1989) might be considered a reasonable nonlinear alternative to capture these asymmetric dynamics. Notably, the maximum likelihood estimates, which are reported in Table 3, show that the transition probabilities are very persistent ($p_{00} = 0.98$, $p_{11} = 0.89$) and that the within-state means are separate from each other ($\mu_0 = 0.32$, $\mu_1 = -1.78$). According to our simulation results, this would help the two-step estimation procedure to compute

[10] In the empirical analysis, we take it as given that the NBER correctly identifies the dates of business cycle turning points.

accurate inferences of the US business cycle dates as Figure 4 actually reveals. This figure, which plots the probabilities that the coincident indicator is in the negative growth rate based on currently available information, shows that the filtered probabilities are in striking accord with the professional consensus as to the history of US business cycles. During periods that the NBER classifies as expansions, the probabilities of recession are usually close to zero. At around the beginning of the NBER-dated recessions the probabilities rise and remain high until around the times the NBER dates the end of the recessions.

In contrast to this two-step procedure, Kim and Yoo (1995), Chauvet (1998) and Kim and Nelson (1998) propose a multivariate dynamic factor model with regime switching in which the two key features of the business cycle are encompassed and estimated in one step. In this alternative model, the numerical maximization of the conditional log likelihood function led to the maximum likelihood estimates that are reported in Table 3 along with their standard errors. Figure 5 plots the nonlinear coincident indicator, which also tracks the business cycle well, with pronounced drops that synchronously correspond to the NBER-designated recessions. In fact, the filtered probabilities that the coincident indicator is in the negative growth rate, which are plotted in Figure 6, also show remarkable success in matching the NBER reference dates.

Interestingly, the dynamics of the estimated coincident indicator from the one-step estimation procedure is in close agreement with the dynamics of the estimated common factor from the linear dynamic factor model. A visual inspection of Figures 3 and 5 suggests that the similarity between the two is striking since they move together synchronously, particularly over business-cycle horizons. The comparative ability of the model to reproduce the US business cycle dates can also easily be evaluated by a visual inspection of Figures 4 and 6, which plot their respective estimated filtered probabilities of recessions. These similarities of the filtered probabilities are also associated with similarities in the estimated coefficients for both models as shown in Table 3.

Although the filtered probabilities seem to indicate that both estimation procedures reproduce the NBER chronology very closely, Table 4 displays the QPS statistics to formally evaluate the performance of the models.[11] The entries of the table reveal the

[11] In the empirical analysis, QPS is defined as in the Monte Carlo analysis with $M = 1$.

high performance score of both models ($QPS_I = QPS_{II} = 0.048$), which implies that there is similar correspondence between the probabilities inferred from the models and the business cycle realizations.

Since the previous results are computed as averages over the entire sample, the analysis may fail to identify the ability of the forecasting models to evaluate the odds of the occurrence of important events such as the turning point dates. To gain some insight into how quickly the last business cycle turning points are identified, the probability scores are also computed on a restricted sample that only includes the first month of each expansion and the first month of each recession. The results are also displayed in Table 4. In this reduced sample, the highest performance score is now achieved by the full Markov-switching dynamic-factor model which assesses QPS reductions of about 20% ($QPS_{II} = 0.535$ versus $QPS_I = 0.453$).

The improvements on business cycle performance of the one step procedure are even clearer when the analysis is restricted to the first month after the recessions (next month after the trough)[12] In this cae, the reductions are of almost 30% ($QPS_{II} = 0.50$ versus $QPS_I = 0.36$). This result can be connected with the literature that finds rapid growth in the recoveries and that makes it difficult to compute accurate business cycle inferences at the very beginning of expansions. Examples are Sichel (1994), Kim and Nelson (1999), and Morley and Piger (2006).

Additionally, the role of the number of indicators used to compute inferences emerges in Figure 7, which plots the filtered probabilities of negative growth for each of the four coincident indicators. Although the figure captures the ability of the individual component indicators to track the business cycle, their performance is inferior to that of the coincident index, as shown in Table 4. Hence, enlarging the set of indicators used to compute inferences from $N = 1$ to $N = 4$, implies a gain in QPS that oscillates between 30% in the case of IP to a 65% in the case of INC. Accordingly, moving to a multivariate framework enables more precise tracking of the cycle.

[12] The differences between recessions and the beginning of the expansions are the highest in the business cycle as documented in papers related to the Friedman´s plucking model, Kim and Nelson (1999), the Third Phase of the business cycle, Sichel (1994) or the bounce back effect, Morley and Piger (2006).

Finally, to assess the actual empirical reliability of the models, their real-time performance in tracking the US business cycles is now evaluated by using a real-time dataset. That is, the inferences are computed monthly over the past 35 years by using only data that would have been available at the month being considered. This is accomplished by estimating the models recursively with new data vintages and evaluating the evidence for a new turning point at the one-period ahead forecast in every period, following the examples of Chauvet and Piger (2008). This method provides a more realistic assessment of how the model would have performed, as it does not assume the knowledge of data revisions that were not available at the time the model would have been used. The sample has to be reduced due to the availability of data and the real-time forecasts are for the period 1976.10-2010.11.

Figures 8 and 9 show that the real-time results are of the same nature as the in-sample results. Overall, the two-step and the one-step procedures exhibit similar forecasting accuracy. However, there is a reduction of about 5% in QPS when the analysis is restricted to examining the ability of the models to compute turning points inferences from the one-step method with respect to the two-step method. As in the in-sample analysis, this gain is concentrated in the recoveries since the reductions in QPS are of more than 40% when analysis is restricted to the periods right after the troughs. According to the theory, the abruptness of these changes in the recoveries is in contrast to the misspecification described in the case of the two-step procedure.[13]

Again, the overall relative real-time performance of multivariate models with respect to univariate models reveals that the former outperform the latter. According to Table 4, the QPS increase from 0.09 in the case of multivariate models to 0.08, 0.12, 0.16, and 0.09 in the case of univariate Markov-switching models of IP, EMPL, INC, and SALES, respectively. Hence, the real-time gains of mixing information from several economic indicators are of the same order of magnitude as the in-sample gains. Accordingly, there is definitely a gain in mixing information even when the analysis accounts for the problems associated with data revisions and real-time forecasting.

[13] We acknowledge that the real-time analysis is developed with five recessions only. However, the analysis helps us not to compute formal inferences but to illustrate the results obtained in the simulations.

5 Conclusions

Business cycle probabilities may be inferred from a set of economic indicators that exhibit comovements and business cycle asymmetries. We show that mixing information helps to fit the nonlinear dynamic behavior even in a real-time out-of-sample analysis. When analyzing the way of mixing information, there are two options in the literature. The first is to compute Markov-switching probabilities from a coincident indicator which is the outcome of a linear dynamic factor model. The second is to compute the probabilities directly from a full Markov-switching dynamic-factor model. Although the first is easier to implement in real time, the second is conceptually more appealing. We examine in this paper the circumstances under which one is preferred to the other.

We point out that the full Markov-switching dynamic-factor model exhibits higher business cycle performance, specially in the turning points. However, we show that the larger the quality of the business cycle indicators used in the analysis, the closer the ability of the two approaches to track the business cycles. This implies that, when the set of indicators included in the analysis are good indicators of the business cycle, the overall differences between the two approaches diminish considerably.. In this case, the superior ability of the full Markov-switching dynamic-factor model appears in the turning points only. In addition, we also show that the more variables that we include in the model, the better the fit. However, the improvements of adding new indicators in terms of business cycle identification dramatically decrease when the number of indicators already used becomes large, even in the case of incorporating indicators of high quality. This implies that the gains from using large sets of indicators can be deceptively low compared with the computational complexity of dealing with nonlinear models that use many indicators.

Using the four constituent indicators of the Stock and Watson (1991) coincident index, production, employment, income and sales, we examine the empirical performance of the models to assess the US business cycle. Since these indicators are of very high quality, we expected a high business cycle performance of the two methods. According to this prior, we show that the coincident indicators obtained from the two alternative estimation procedures track the business cycle well, with obvious and pronounced drops corresponding to the NBER-designated recessions. In addition, the business cycle dates obtained from their respective filtered probabilities are in both cases almost identical to the officially recognized US business cycle chronology. However, we obtain some relative gains of the full Markov-switching dynamic-factor model when the analysis is focused on turning points detection. We confirm this result in a real-time analysis of the models' performance over the past 35 years.

6 Appendix

Proof of Proposition 1:

The conditional distribution of the N-th indicator , $f(y_{N,t}|s_t = i, I_{1,t})$, is given by

$$f(y_{N,t}|s_t = i, I_{1,t}) = \frac{1}{\sqrt{2\pi\sigma^2_{N|1}}} \exp\left(-\frac{1}{2\sigma^2_{N|1}}\left(y_{N,t} - y^{(i)}_{N,t|t}\right)^2\right), \tag{A1}$$

where $y^{(i)}_{N,t|t}$ and $\sigma^2_{N|1}$ are its mean and variance which can be derived by using the well-known expressions for the conditional first two moments of a multivariate normal random vector. Let $\mathbf{y}_{(N-1),t} = (y_{1,t}, ..., y_{N-1,t})'$ be the vector of the $N-1$ first observed series, let $\mathbf{\Sigma}_{N1} = cov(y_{N,t}; \mathbf{y}_{(N-1),t}|s_t = i)$ be the $1 \times (N-1)$ vector of conditional covariances between $y_{N,t}$ and the elements of the vector $\mathbf{y}_{(N-1),t}$, let $\mathbf{\Sigma}_{11} = var(\mathbf{y}_{(N-1),t}|s_t = i)$ be the $(N-1) \times (N-1)$ conditional covariance matrix of $\mathbf{y}_{(N-1),t}$, let $\widetilde{\mathbf{\Lambda}} = (\lambda_1, ..., \lambda_{N-1})'$ be the $(N-1) \times 1$ vector of factor loadings associated with the elements of the vector $\mathbf{y}_{(N-1),t}$, and let $\widetilde{\mathbf{\Sigma}}_u = diag(\sigma^2_1, ..., \sigma^2_{N-1})$ be the $(N-1) \times (N-1)$ diagonal covariance matrix associated with the observation equation for the first $N-1$ variables as well. Taking into account that

$$\mathbf{\Sigma}_{N1} = \sigma^2_a \lambda_N \widetilde{\mathbf{\Lambda}}', \tag{A2}$$

and that

$$\mathbf{\Sigma}_{11} = \sigma^2_a \widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{\Lambda}}' + \widetilde{\mathbf{\Sigma}}_u, \tag{A3}$$

one can use the expression for the inverse of the sum of two matrices to compute the inverse of $\mathbf{\Sigma}_{11}$ as

$$\begin{aligned}\mathbf{\Sigma}_{11}^{-1} &= \widetilde{\mathbf{\Sigma}}_u^{-1} - \widetilde{\mathbf{\Sigma}}_u^{-1}\widetilde{\mathbf{\Lambda}}\left(\widetilde{\mathbf{\Lambda}}'\widetilde{\mathbf{\Sigma}}_u^{-1}\widetilde{\mathbf{\Lambda}} + \frac{1}{\sigma_a^2}\right)^{-1}\widetilde{\mathbf{\Lambda}}'\widetilde{\mathbf{\Sigma}}_u^{-1} \\ &= \widetilde{\mathbf{\Sigma}}_u^{-1} - \frac{1}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1}\frac{\lambda_i^2}{\sigma_i^2}}\widetilde{\mathbf{\Sigma}}_u^{-1}\widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{\Lambda}}'\widetilde{\mathbf{\Sigma}}_u^{-1}. \end{aligned} \tag{A2}$$

Hence, the conditional mean $y_{N,t|t}^{(i)}$ can be expressed as

$$\begin{aligned} y_{N,t|t}^{(i)} &= E(y_{N,t}|s_t = i, I_{1,t}) = E(y_{N,t}|s_t = i, y_{1,t}, ..., y_{N-1,t}) \\ &= E(y_{N,t}|s_t = i) + \mathbf{\Sigma}_{N1}\mathbf{\Sigma}_{11}^{-1}(\mathbf{y}_{(N-1),t} - E(\mathbf{y}_{(N-1),t}|s_t = i)) \\ &= \lambda_N\mu_i + \sigma_a^2\lambda_N\widetilde{\mathbf{\Lambda}}'\left(\widetilde{\mathbf{\Sigma}}_u^{-1} - \frac{1}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1}\frac{\lambda_i^2}{\sigma_i^2}}\widetilde{\mathbf{\Sigma}}_u^{-1}\widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{\Lambda}}'\widetilde{\mathbf{\Sigma}}_u^{-1}\right)(\mathbf{y}_{(N-1),t} - \widetilde{\mathbf{\Lambda}}\mu_i) \\ &= \lambda_N\left(\mu_i + \frac{1}{\frac{1}{\sigma_a^2} + \sum_{j=1}^{N-1}\frac{\lambda_j^2}{\sigma_j^2}}\widetilde{\mathbf{\Lambda}}'\widetilde{\mathbf{\Sigma}}_u^{-1}(\mathbf{y}_{(N-1),t} - \widetilde{\mathbf{\Lambda}}\mu_i)\right), \\ &= \lambda_N f_{t|t}^{(i)}, \end{aligned} \tag{A3}$$

where

$$f_{t|t}^{(i)} = \left(\mu_i + \frac{1}{\frac{1}{\sigma_a^2} + \sum_{j=1}^{N-1}\frac{\lambda_j^2}{\sigma_j^2}}\widetilde{\mathbf{\Lambda}}'\widetilde{\mathbf{\Sigma}}_u^{-1}(\mathbf{y}_{(N-1),t} - \widetilde{\mathbf{\Lambda}}\mu_i)\right). \tag{A4}$$

The conditional variance, $\sigma_{N|1}^2$, can be expressed as

$$\begin{aligned} \sigma_{N|1}^2 &= var(y_{N,t}|s_t = i, I_{1,t}) = var(y_{N,t}|s_t = i, y_{1,t}, ..., y_{N-1,t}) = \\ &= var(y_{N,t}|s_t = i) - \mathbf{\Sigma}_{N1}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{1N} \\ &= \lambda_N^2\sigma_a^2 + \sigma_N^2 - \sigma_a^4\lambda_N^2\widetilde{\mathbf{\Lambda}}'\left(\widetilde{\mathbf{\Sigma}}_u^{-1} - \frac{1}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1}\frac{\lambda_i^2}{\sigma_i^2}}\widetilde{\mathbf{\Sigma}}_u^{-1}\widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{\Lambda}}'\right)\widetilde{\mathbf{\Lambda}} \\ &= \lambda_N^2\sigma_a^2 + \sigma_N^2 - \sigma_a^4\lambda_N^2\left(\sum_{i=1}^{N-1}\frac{\lambda_i^2}{\sigma_i^2} - \frac{\left(\sum_{i=1}^{N-1}\frac{\lambda_i^2}{\sigma_i^2}\right)^2}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1}\frac{\lambda_i^2}{\sigma_i^2}}\right) \\ &= \sigma_N^2 + \frac{\lambda_N^2}{\frac{1}{\sigma_a^2} + \sum_{i=1}^{N-1}\frac{\lambda_i^2}{\sigma_i^2}}. \end{aligned} \tag{A5}$$

Finally, then the KL divergence is given by

$$
\begin{aligned}
KL &= \int \ln \frac{f(y_{N,t}|s_t=1, I_{1,t})}{f(y_{N,t}|s_t=0, I_{1,t})} f(y_{N,t}|s_t=1, I_{1,t}) dy_{N,t} \\
&= \frac{1}{2\pi\sigma^2_{N|1}} \int \left(\left(y_{N,t} - \lambda_N f^{(0)}_{t|t} \right)^2 - \left(y_{N,t} - \lambda_N f^{(1)}_{t|t} \right)^2 \right) f(y_{N,t}|s_t=1, I_{1,t}) dy_{N,t} \\
&= \frac{1}{2\sigma^2_{N|1}} \int \left[2y_{N,t}\lambda_N \left(f^{(1)}_{t|t} - f^{(0)}_{t|t} \right) + \lambda^2_N \left(\left(f^{(0)}_{t|t} \right)^2 - \left(f^{(1)}_{t|t} \right)^2 \right) \right] f(y_{N,t}|s_t=1, I_{1,t}) dy_{N,t} \\
&= \frac{1}{2\sigma^2_{N|1}} \left[2\lambda^2_N f^{(1)}_{t|t} \left(f^{(1)}_{t|t} - f^{(0)}_{t|t} \right) + \lambda^2_N \left(\left(f^{(0)}_{t|t} \right)^2 - \left(f^{(1)}_{t|t} \right)^2 \right) \right] \\
&= \frac{\lambda^2_N}{2\sigma^2_{N|1}} \left[\left(f^{(1)}_{t|t} \right)^2 - 2 f^{(1)}_{t|t} f^{(0)}_{t|t} + \left(f^{(0)}_{t|t} \right)^2 \right] \\
&= \frac{\lambda^2_N}{2\sigma^2_{N|1}} \left(f^{(1)}_{t|t} - f^{(0)}_{t|t} \right)^2 . \qquad \text{(A6)}
\end{aligned}
$$

Using the expressions of $\sigma^2_{N|1}$ and $f^{(i)}_{t|t}$, one can easily obtain expression (20).

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Table 1. Two-step versus one-step estimation procedures

σ^2	One step				Two steps			
	μ_0-μ_1							
	1	2	4	10	1	2	4	10
0.5	0.201 (0.332)	0.097 (0.255)	0.016 (0.070)	5.09E-08 (1.29E-08)	0.233 (0.354)	0.098 (0.264)	0.0453 (0.245)	0.013 (0.089)
1.5	0.202 (0.326)	0.124 (0.275)	0.027 (0.095)	4.22E-06 (2.67E-05)	0.246 (0.356)	0.131 (0.296)	0.055 (0.276)	0.019 (0.120)
4.5	0.204 (0.320)	0.171 (0.307)	0.061 (0.169)	4.02E-04 (1.47E-03)	0.2897 (0.366)	0.211 (0.344)	0.088 (0.331)	0.031 (0.186)

Notes. Entries show the average over the replications of the averaged squared deviation of filtered probabilities of low-mean state from the 1000 generated business cycle sequences. The results when the analysis restricted to the first month after phase shifts are in parentheses. "One step" refers to a MS-DFM and "two steps" refers to a MS to a linear common. The replications use three indicators (N=3).

Table 2. The role of N on MS-DFM

σ^2	N			
	1	3	5	7
0.5	0.122	0.097	0.092	0.088
1.5	0.168	0.124	0.109	0.099
4.5	0.202	0.171	0.141	0.124

Notes. Entries show the average over the replications of the averaged squared deviation of filtered probabilities of low-mean state from the 1000 generated business cycle sequences. N denotes the number of variables included in the model, and σ^2 the variance of the idiosyncratic shocks. The model has been generated with μ_0-μ_1=2

Table 3. Maximum likelihood estimates

Two-step procedure						
		Factor	IP	Empl	Inc	Sales
DMF	λ_i	-	0.69 (0.03)	0.50 (0.03)	0.28 (0.03)	0.45 (0.03)
	ϕ_1	0.47 (0.06)	-0.26 (0.07)	0.23 (0.03)	-0.20 (0.02)	-0.36 (0.04)
	ϕ_2	0.22 (0.05)	-0.21 (0.08)	0.53 (0.04)	-0.05 (0.04)	-0.15 (0.05)
	σ_i^2	1	0.26 (0.04)	0.24 (0.02)	0.85 (0.03)	0.59 (0.03)
MS		μ_1	μ_2	$\sigma_{a^*}^2$	p_{00}	p_{11}
		0.32 (0.04)	-1.78 (0.14)	0.80 (0.05)	0.98 (0.01)	0.89 (0.04)
One-step procedure						
Indicators			IP	Empl	Inc	Sales
	λ_i		0.69 (0.03)	0.42 (0.02)	0.28 (0.04)	0.46 (0.03)
	ϕ_1		-0.18 (0.08)	0.24 (0.03)	-0.20 (0.02)	-0.34 (0.04)
	ϕ_2		-0.16 (0.08)	0.54 (0.04)	-0.05 (0.04)	-0.15 (0.05)
	σ_i^2		0.26 (0.04)	0.27 (0.02)	0.85 (0.03)	0.57 (0.03)
Factor		μ_1	μ_2	$\sigma_{a^*}^2$	p_{00}	p_{11}
		0.32 (0.07)	-2.00 (0.20)	1	0.98 (0.01)	0.85 (0.05)

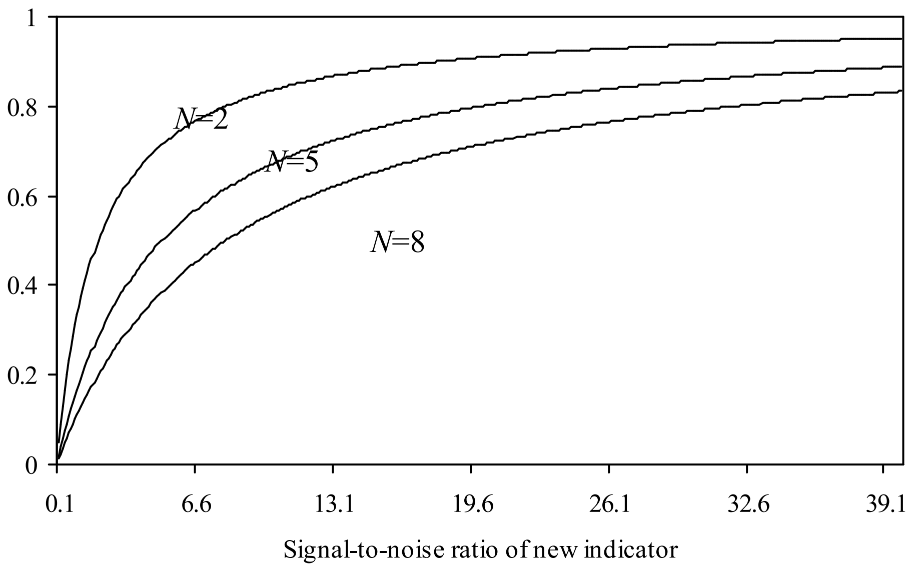
Note: Standard errors are in parenthesis.

Table 4. Empirical performance

	1-step	2-steps	IP	Empl	Inc	Sales
In sample (1967.01-2010.11)						
Total	0.05	0.05	0.07	0.12	0.13	0.09
Turning points	0.31	0.39	0.54	0.67	0.42	0.33
Real time (1976.10-2010.11)						
Total	0.06	0.06	0.08	0.12	0.16	0.09
Turning points	0.51	0.54	0.49	0.73	0.39	0.36

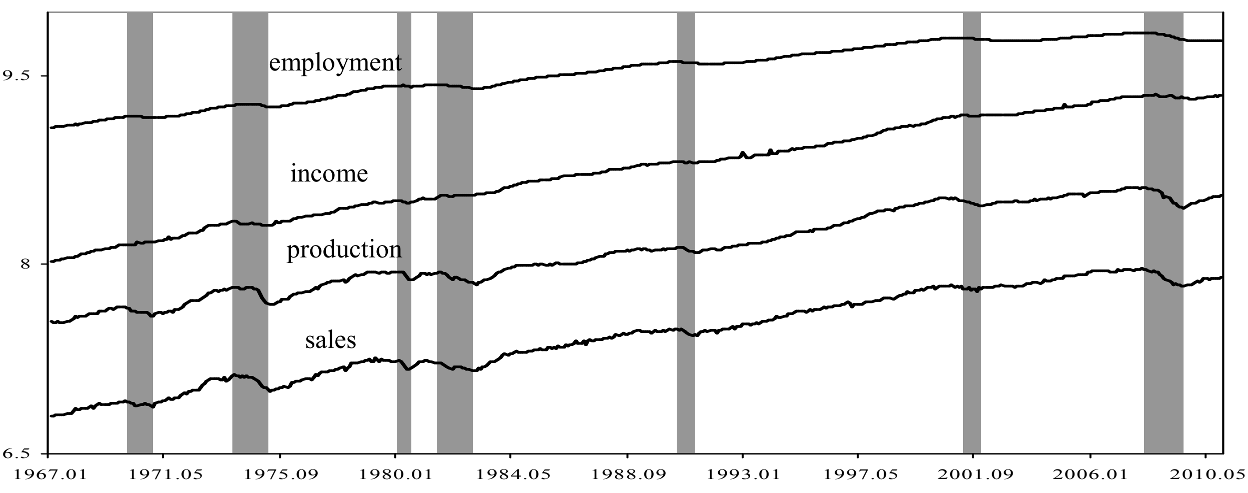
Note. Entries labeled as “total” refer to QPS statistics. In the case of entries labeled as “turning points”, the QPS is computed by using the first month after the phase shifts.

Figure 1. KL divergence



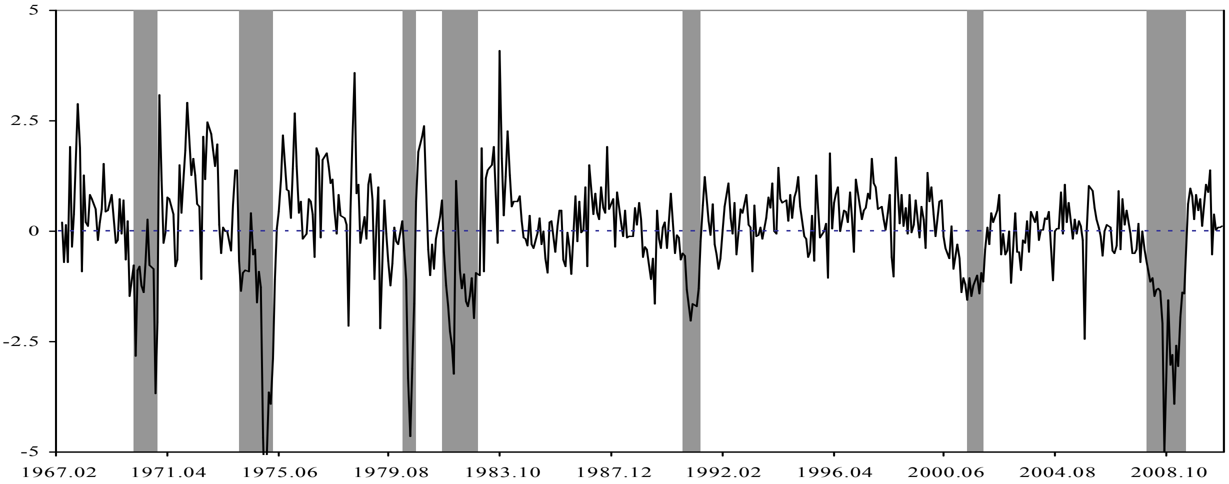
Note. This figure measures the gains of adding a new indicator (the initial set contains *N*-1 indicators) to infer the probability of recession when a recession occurs as a function of its signal-to-noise ratio.

Figure 2. Logs of monthly economic indicators



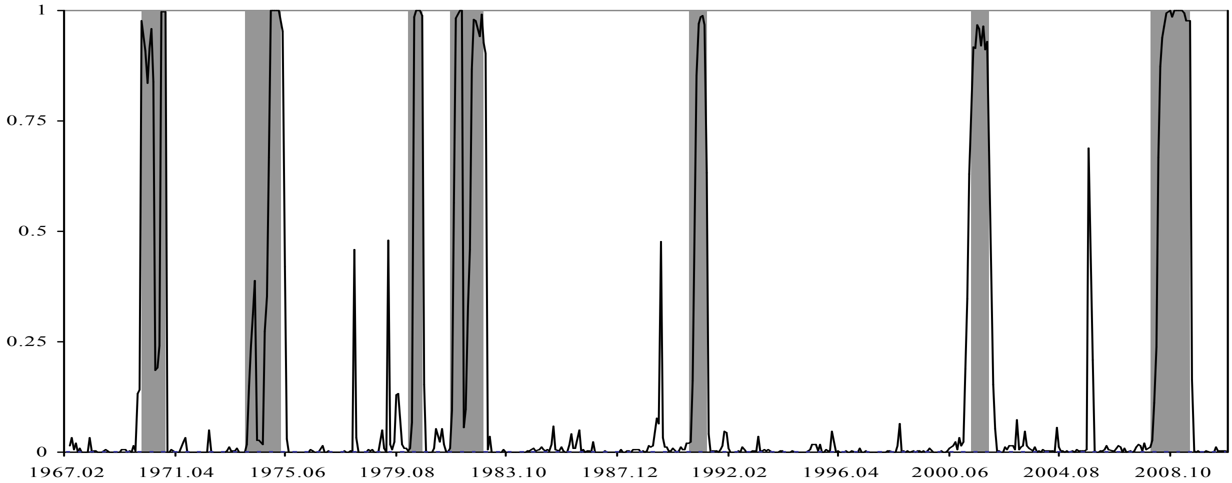
Notes: Constants were added to facilitate graphing. Shaded areas correspond to recessions as documented by the NBER.

Figure 3. Common factor from linear DFM



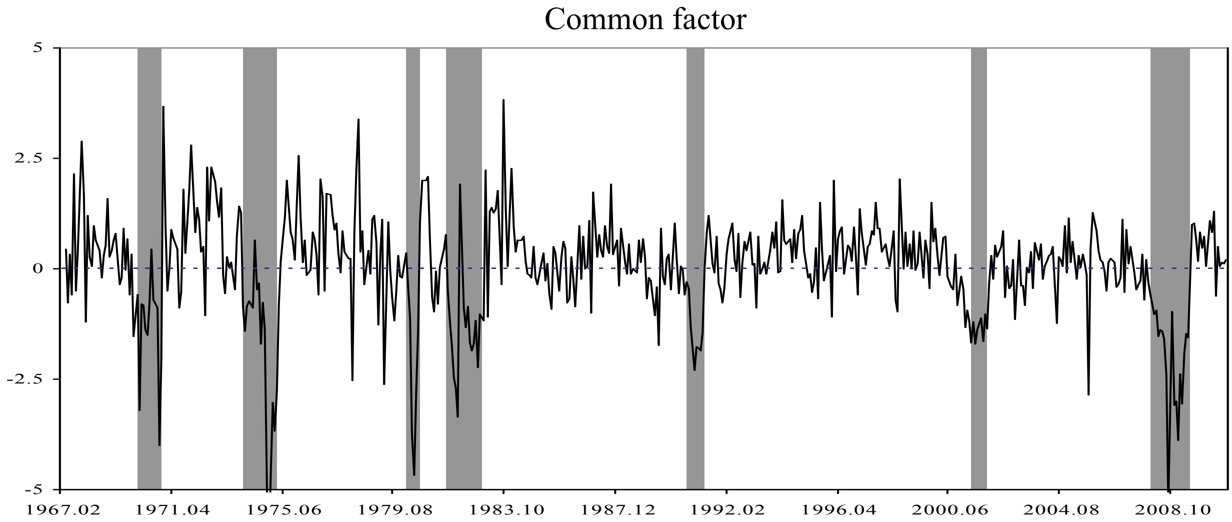
Notes: Shaded areas correspond to recessions as documented by the NBER.

Figure 4. Filtered probabilities from linear common factor



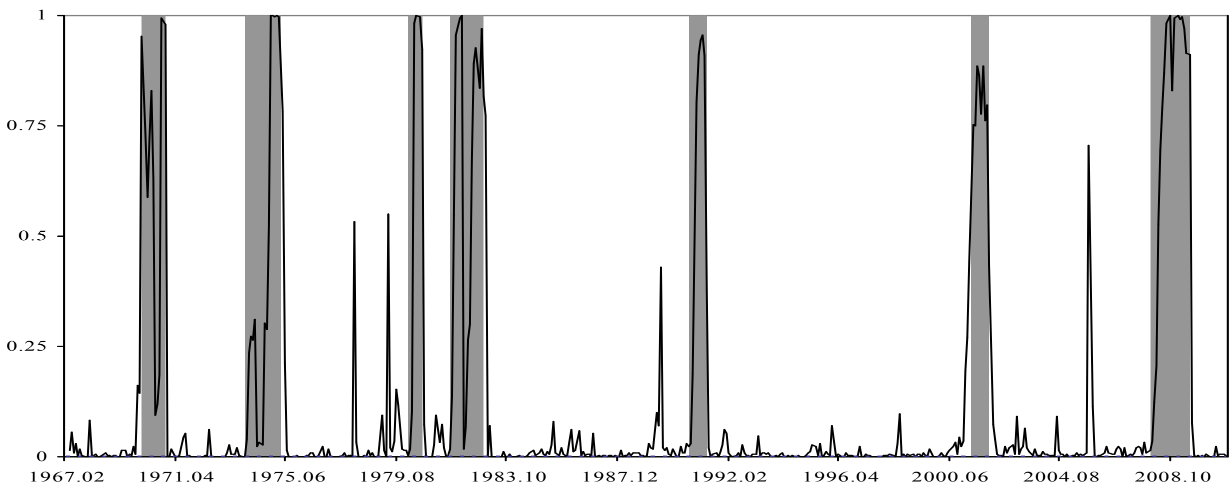
Notes: Shaded areas correspond to recessions as documented by the NBER.

Figure 5. Common factor from MS-DFM



Note: Shaded areas correspond to recessions as documented by the NBER.

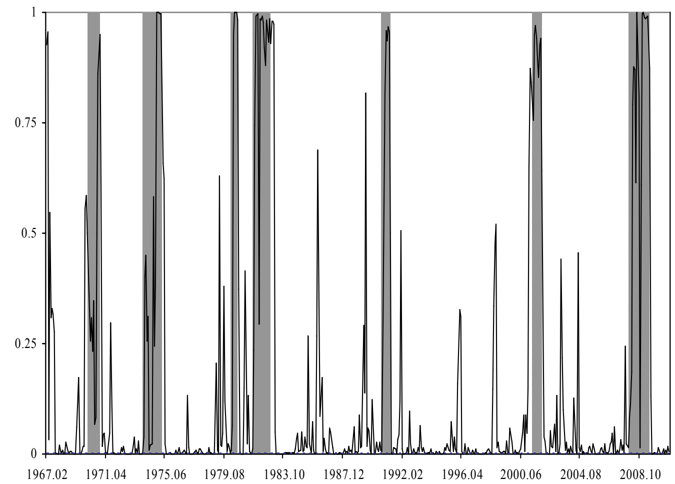
Figure 6. Filtered recession probabilities from MS dynamic factor



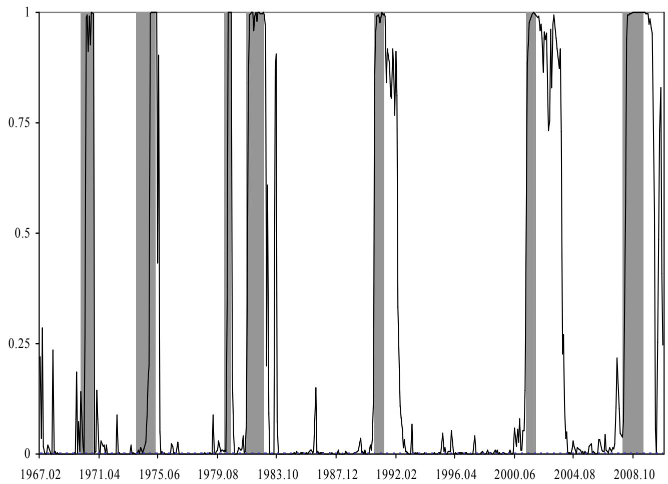
Note: Shaded areas correspond to recessions as documented by the NBER.

Figure 7. Filtered recession probabilities from each indicator

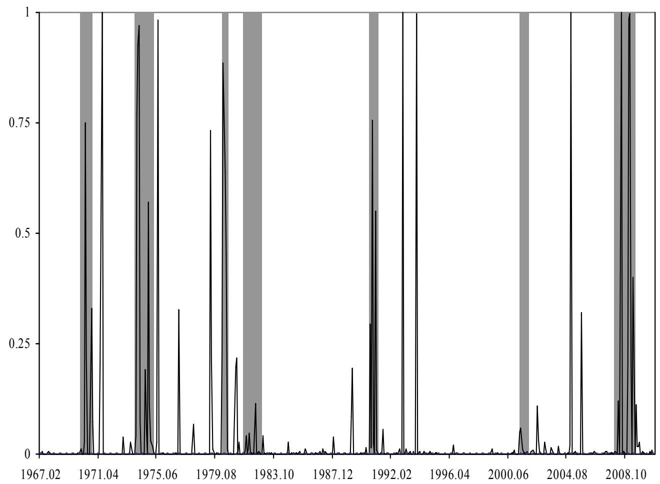
Industrial production



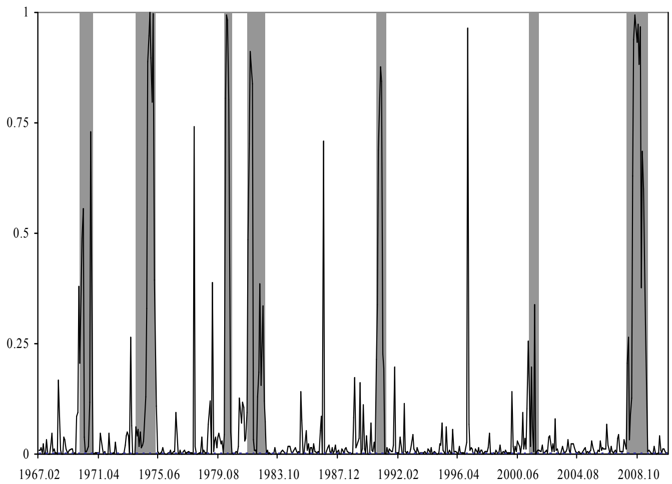
Employment



Income

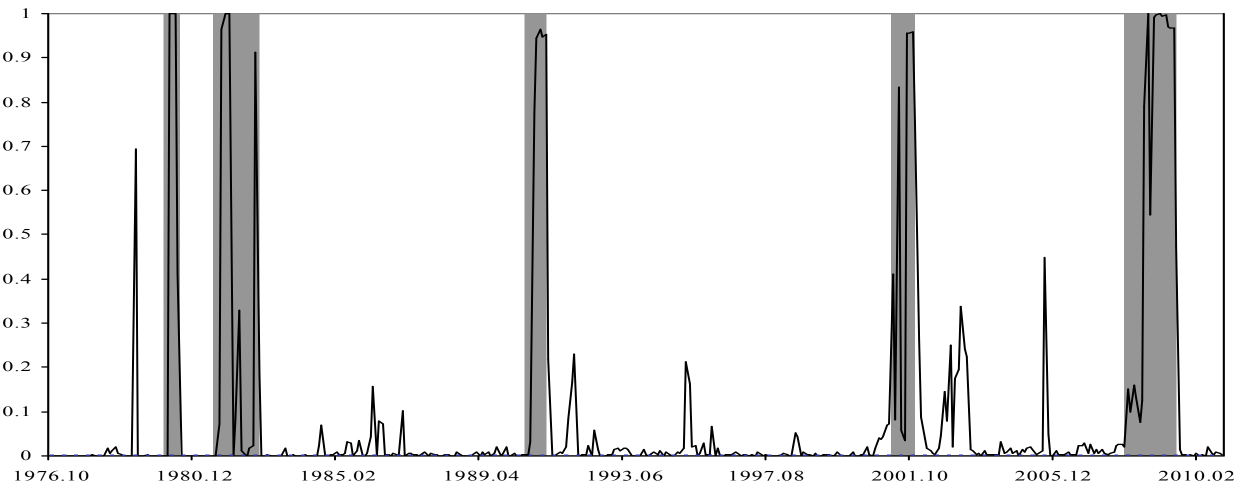


Sales



Note: Shaded areas correspond to recessions as documented by the NBER.

Figure 8. Real-time one-period-ahead forecasts from MS-DFM



Note: The graph shows the one-period ahead out of sample forecast of the probability of being in recession. Shaded areas correspond to recessions as documented by the NBER.

Figure 9. Real-time one-period-ahead forecasts from linear common factor