Trip-to-Work Congestion in a Network with Two Imperfectly Substituted Modes*

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Abstract

This paper is concerned with the welfare properties of a rational microeconomics model on traffic congestion for an existing infrastructure and modal system. Maximizing commuter behavior may result in undesirable levels of congestion, an externality that increases travel time and travel costs and, hence, reduces welfare. The main contributions of this paper are the following. First, we provide an acute definition of traffic congestion treated as an externality. Second, we determine the Pareto efficient level of congestion; in particular, we find a threshold, which depends on the infrastructure level and the number of users, that distinguishes two regions: one region without congestion, where decentralized allocations are Pareto efficient, and another region with congestion, where the decentralized individual allocations differs from the centralized ones. Third, there may exist a social optimum congestion level. Finally, we present an unifying environment to study efficiency for transportation policy implementation.

Key words: Congestion, Externality, Pareto optimality, Transport, Urban Congestion, External costs, Modal share, Transport policy mechanisms

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1 Introduction

Motivation Traffic congestion is considered as one of the greatest problems in cities, which daily affects citizens moving within towns. It recurrently appears at certain time slots of the day, mainly determined by the working schedule. Local authorities are aware about this issue, and they keep rethinking new policies to alleviate intercity congestion. Build a new infrastructure, or even improving the existing ones, used to be expensive in terms of public resources and takes a long; they are more popular in respect with less appealing political measures that affects commuters' behaviour despite cheaper and easier to implement. However, although traffic congestion is a negative externality, it is not clear that removing or even reducing traffic congestion be a social efficient policy due to the implementation of a transportation policy could result in benefits lower than costs for city commuters. Consequently it is relevant for local policymakers to achieve the optimal allocation of commuters among travel modes at minimum congestion cost; that is, to find the optimal level of congestion. It can vary from city to city, and depends on the infrastructures available, alternative modes of transportation and city commuters preferences.

Goal The purpose of this paper is to determine the aggregate level of traffic in a city as the result of individual modal decisions when commuters travel for working purposes at an inelastic peak-hour schedule.

Model To do this, we make use a microeconomic decision link flow model of the traffic stream where commuters must undertake a trip from the suburbs to the city center along some existing road infrastructure, which presents particular features (e.g., capacity, number of lanes, etc.). Trips are undertaken for working purposes, so commuters are restricted to travel at some slot of the day, denoted by "trip to work peak-hour," and consequently there is no possibility to divert trips to off-peak slots. Two imperfectly substituted transportation modes are available: own car and collective bus, where congestion interaction between both modes exists. Commuters have exogenous monetary and time endowment, as well as they all are one-seater car owners. Time is devoted to leisure and/or travel activities; the exogenous monetary resources are spent consuming a good inelastically supplied by the productive sector, and on travel expenses. In addition, commuters enhance welfare by consuming and developing leisure activities, but they dislike traveling: trip time, whether by car or bus, reduces their welfare, and the uncomfortable trips are additionally qualified by the transportation mode they travel. The mode chosen and the time traveling are input variables to produce the idiosyncratic bad. Heterogeneity among commuters regards their valuation of the trip time, and we will assume that the distribution of these valuations is uniform.

Transportation modes require monetary and temporal resources: monetary cost is fixed for buses, while cars costs more the more trip time. With respects of temporal resources, they will depend on the features of the infrastructure, the transportation modal system, and the amount of vehicles in road. We assume a free-flow trip time for cars and buses traveling along the link when no congestion exists. We define traffic flow as the total number of vehicles on roads at the peak period; that is, the total number of individual cars plus the total number of buses. Consequently, each commuter's trip time spent traveling in some transportation mode along a particular road is the output of a technology that makes use free-flow trip time, gathering the exogenous infrastructure and modal features, and the traffic flow as inputs. Whenever the traffic flow increases beyond certain threshold, it will affect commuters' travel time and become a negative externality, called traffic congestion.

We first analyze the social planner problem, and characterize the Pareto optimal level of traffic flow, as well as the resulting level of the congestion externality, if any. Next, we present the decentralized equilibrium.

The literature. There is a vast literature in engineering and economics on traffic congestion dealing with policy prescriptions for reducing or removing congestion, most of them aimed to empirical research.[1] The economic literature treats the misallocation inherent in roads when private rather than social costs are considered by a commuter to take his travel decision. This justify the introduction of congestion tolling to equalize each commuter's private and marginal social costs.[2] However, the optimal toll is determined by using an "ad hoc" demand for traffic function and "ad hoc" private and social cost traffic functions, which both lack any microeconomics foundation. As Arnott (2000) critiques, in this literature the congestion function is treated as a technological datum, and do not incorporate commuters' behaviour. In contrast, our paper explicitly presents the behavior of commuters, who are affected by the congestion externality. Hence, the Pareto-optimal and decentralized levels of traffic, and then the congestion level, can be obtained in a unified rational microeconomics equilibrium set-up.

The closest works to ours are Marchand (1968), Sherman (1971) and more recently Parry (2002), who are few exceptions in the literature by considering the congestion issue on individual travel decision grounds (see table 1). Marchand was the first to derive explicitly the optimal level of congestion within a general equilibrium model. Sherman extended Marchand's work in two ways: transportation modes contribute to each other's congestion, and there is an imperfect substitutability between modes by considering that commuter preferences are differently affected when traveling in alternative modes. Both papers consider heterogeneity of agents on income and preferences (on consumption and number of trips), and congestion increases monetary trip costs, lowering consumption and the number of trips.

With regards to Parry (2002), he compares numerically the economic efficiency

[1] For an overview see Lindsey and Verhoef (2000a,b) and Arnott (2001).

[2] The literature used to consider "congestion" as a problem, something to solved up. It mainly focus on overcoming the congestion externality by finding second-best policies for welfare improving assessments under different tax policies (see Lindsey and Verhoef, 2000a, b, Parry and Bento, 2000, or Kveiborg, 2001). Consequently, congestion is treated as a market failure that results in an inefficient allocation, despite no theoretical foundation supports this statement.

	Marchand(68)	Sherman(71)	Parry(02)	Model
Subst. modes	Perfect	**Imperf. (prefer.)**	**Imperf. (prefer.)**	**Imperf. (prefer.)**
Cong.interact. (modes)	NO (peak-hour)	**YES** (peak-hour)	NO (peak-hour& off)	**YES (peak-hour)**
Heterogeneity:				
•Prefer.	c(+) n.trips(+)	c(+) n.trips(+)	c(+),l(+) n.trips(+)	c(+),l(+) **trip time(−)**
•Monetary	**income**	**income**	wages	**income**
Temp.const.	NO	NO	**YES**	**YES**
Congestion technology	increases: expend.	increases: expend.	increases: **expend., time**	increases: **expend., time**

Table 1: Main differences and analogies with close literature

of alternative transportation policy measures to reduce congestion, using a general equilibrium model. The environment is similar to Marchand's and Sherman's, but considering a time constraint and that commuters can undertake both peak-hours trips, on several transportation modes, or off-peak trips. It shares Marchand's no interaction of modes in terms of congestion (Parry explicitly excludes traveling by bus), as well as Sherman's imperfect substitutability among modes. Agents are only heterogeneous in wages, and congestion increases monetary and time costs, affecting consumption, leisure, labor time and the number of trips.

In the present paper we consider that transportation modes are imperfect substitutes for commuters by considering that preferences depends on the transportation mode chosen, analogous to Sherman and Parry. We also share Sherman's consideration of the congestion interaction between modes, and commuters' heterogeneity of preferences and income. Besides, our model consider explicitly a time constraint, as in Parry.

Our model differs from these works in three ways that will be shown to be essential. First, we consider that time involved in travel activities decreases welfare for commuters. Transportation is a required intermediate good to allow individuals to obtain monetary resources, so travel trips are a medium to achieve this goal, a waste of individual's time and resources. Accordingly, we do not consider transportation as a good that increases commuters welfare. Second, traffic congestion affects commuters not only by increasing monetary and time costs, like in Parry (2000), but it also affects negatively on commuter preferences due to congestion increases trip time. Finally, we focus on peak-hour trips, like in Marchand and Sherman, so Parry's possibility to transfer trips to off-peak slots does not exist.

In addition, and more important, this paper aims to solve the internal inconsistency of the way the above authors model the congestion externality. Marchand (1968), Sherman (1971) and Parry (2002) treat congestion as a kind of *aggregate good* along the period considered, for example, a day, a month or a year. Their commuters choose the number of trips to undertake in each of the different transportation modes at the period considered, and the total number of trips of all commuters determines the traffic flow. Then traffic congestion is defined as the influence of traffic flow on travel time, and the "peak-hour congestion" can be interpreted as the ratio of aggregate congestion over the number of times the peak-hour slot is reproduced within the period. However, it is not clear why a commuter chooses several transportation modes that all affects simultaneously the peak-hour congestion in so short period of time. One possibility is that each commuter takes different modal choices along each shorter peak-hour slots through the period. However new difficulties arise. First, it is not clear why repeated environments results in different modal choices. Second, due to commuters choose to undertake trips in different modes, it is difficult to understand in this set-up why different commuter allocations between alternative modes result in the same peak-hour congestion unless a unrealistic assumption on overall commuters coordination is realized. In addition, in Sherman's case where congestion interaction among modes exists, the commuter's decision to undertake a number of trips by car will affect the (aggregate) congestion which interacts with the number of trips he decides to undertake by bus, and vice versa. But, this two-direction interaction can only be possible if commuter undertakes simultaneously several modal trips which is no realistic.

These problems stem on two facts. First, commuters are allowed to travel among several modes and, as the period of time is shrunk, it is difficult to understand why a commuter takes different modal choices under identical network environment and preferences. Unlike these authors, in our model each commuter can only choose one transportation mode to trip at a specific slot of time of the day where congestion may occur ("peak-hour"). Second, and conceptually more relevant, these authors do not provide a clear definition of congestion, but a vaguely description of its effects.[3]

Our contribution. The main contributions of this paper are the following. First, we provide an acute definition of traffic congestion treated as an externality. Second, we study whether the level of traffic becomes a congestion externality or not, so we analyze if a certain level of congestion may be socially optimal, relying on transportation and road network features, as well as on citizens characteristics. In particular, we find a threshold, which depends on the infrastructure level and the number of users, that distinguishes two regions: one region without congestion, where decentralized allocations are efficient, and another region with congestion, where the decentralized individual allocations differs from the centralized ones. In addition, the congestion externality may result in a non-Pareto efficient decentralized allocation, as the previous literature seems to point out, and second-best policies are required to improve social welfare. An interesting feature of our model is that it allows us to gauge the magnitude of this inefficiency. Third, we present an unifying environment to study efficiency for transportation policy implementation.

[3]For example "There is congestion [...] at peak travel time [meaning] that the presence of an extra vehicle slows down the average speed of other drives, hence raising their travel times." (Parry, 2000, p.337).

Road map of the paper This work develops through the following sections. In Section 2 we present a definition of congestion. In Section 3 the primitives of the model. The social planner problem is solved in Section 4. Section 5 displays the decentralized equilibrium and we show that the resulting allocations may not be optimal. Finally, Section 6 summarizes conclusions and indicates further research.

2 A definition of congestion.

Surprisingly, the literature of transportation economics does not display an acute definition of congestion. For example, reference works like Walters (1987) just indicates that the "common usage" of congestion is used to "denote circumstances where there is *some* interaction and slowing of vehicle below their traffic-free speed." (p.571).

Before defining congestion, we must have in mind two issues. First, it has both a spatial and temporal component. In terms of space, it may occur along both short and long sections of roadway, while temporally, it may occur for a few minutes, a few hours or the entire day. Second, there are three key elements in play for congestion to occur: two of them are technological, the features of the infrastructure and the transportation modal system; while the third falls into the economics realm: the modal allocation of commuters resulting from their own choice.

In what respects commuters decisions, we consider that, in Debreu's spirit, commuters demand the commodity "transportation from A to B at a particular period of time." This commodity is produced with a combination of three inputs: i) travel time; ii) one of the existing alternative transportation modes (car, bus, rail, etc.) that makes use; and, iii) one of the existing network with its specific features linking A to B. Next, we display several definitions on traffic flow, traffic free-flow, and finally, congestion.

Definition 1 *Given a number of users, we define* **traffic flow of a particular link at a particular moment of time** *as the number of vehicles simultaneously using it.*

Definition 2 *Given the technological elements, we define* ***traffic free-flow*** *at a particular link referring to the case where the technological combination between travel time and transportation modes is not affected by the number of vehicles using the link.*

Definition 3 ***Congestion at a particular link from A to B at a period of time*** *is present whenever the well-being of every commuter is directly affected by the travel decision of any other commuter disturbing her technology combination between travel time (output) and transportation mode with respect to the traffic free-flow case, and the number of vehicles in the link.*[4]

[4]Our definition is in tune with the workable definition of traffic congestion provided by Lomax,

Consequently, congestion is thought as an externality of all commuters when using the congested link that affects the technological production of the transportation commodity.

3 The model.

Our main concern is to study the mutual vehicle interaction in a given road infrastructure at a particular slot of the day brought by the simultaneous commuters' modal decision. We present a simple model of transportation that gathers the three main features regarding congestion: two of them technological –the infrastructure and the different transportation mode characteristics–, and the third dealing with the individuals modal decision.

3.1 Ingredients: the primitives of the economic model

The main features of the model are the following,

- There is a road infrastructure that links A and B;
- There are a number of commuters $\mathcal{I} \equiv [0, I]$ that travels along the infrastructure; and,
- Two transportation modes are available to undertake the trip: car and collective bus.

The commodities of this economy are the following,

- *leisure time* at a particular slot;
- a *consumption good*; and,
- *transportation services from A to B at a particular slot of time.*

Next, we enumerate several assumptions.

3.1.1 Assumption on the road infrastructure

Assumption 1 *The road infrastructure that links A and B is represented by several given characteristics: capacity, number of lanes, etc. No alternative itinerary, such as back-roads, exists.*

Turner, Shunk *et al* (1997) who defined two terms: "Congestion is travel time or delay in excess of that normally incurred under light or free-flow travel conditions; and, unacceptable congestion is travel time or delay in excess of an agreed-upon norm. The agreed-upon norm may vary by type of transportation facility, travel mode, geographic location and time of day." Of course, "agreed-upon norm" falls into what economists call preferences. There are other definitions such as ECMT (1999) "congestion is the impedance vehicles impose on each other, due to the speed-flow relationship, in conditions where the use of a transport system approaches its capacity."

		bus	car
capacity		B	1
resources	temporal	t_b^i	t_c^i
	monetary	k_b^i	$k_c(t_c^i) = k_c^0 + k_c t_c^i$

Table 2: Transportation mode features.

3.1.2 Assumption on the transportation modes

Car and collective bus, termed as c and b respectively, are the only two transportation modes available that link A and B.

Assumption 2 *Each transportation mode is fully characterized by its capacity in passengers, and by the temporal and monetary resources spent in each.*

2.1. Capacity. *We will assume that only one-seater cars exist, while buses can transport B passengers.*

2.2. Temporal resources. *To reach B from A, the commuter i is required to spent t_c^i units of time when traveling by car, or t_b^i when traveling by bus.*

2.3. Monetary resources. *The bus fare k_b is fixed. In contrast, the cost for car commuters is an increasing convex function of the time spent traveling. We will assume that this function is linear, i.e.,*

$$k_c\left(t_c^i\right) = k_c^0 + k_c t_c^i,$$

where k_c is a constant that includes several costs per unit of travel time such as fuel, tires, breaks, etc.; and k_c^0 represents fixed costs such as the parking fare.

3.1.3 Assumptions on the commuters

There are a number of commuters $\mathcal{I}$, who must undertake a trip along the existing road infrastructure at a particular slot of time.

The trip features

Consumers are compelled to travel from A to B in order to undertake working activities at destiny taking one modal decision.

Assumption 3 *Trips are developed for working purposes, and commuters will suffer an infinite cost whether arriving at work early nor late.*

Assumption 4 *Commuters travel at a particular slot of the day making use only one transportation mode.*

The former assumption brings about traveling activities are undertaken at the slot of time denoted by (*trip-to-work*) *peak-hour*, just before the working timetable. It also entails the impossibility to divert trips to off-peak slots. The assumption 3 means that each commuter takes just one modal discrete choice to attend to work, whether by car or bus. For auxiliary purposes we will introduce two indicatrix variables that can only take two values, one or zero, representing whether commuter i choses or not mode m: $v_m^i \in \{0,1\}$ with $m = c$, b, where at least and at most one mode is used, that is, $v_c^i v_b^i = 0$ and $v_c^i + v_b^i = 1$. Then, the number of cars is given by $c = \int_{i\in[0,\mathcal{I}]} v_c^i \, di$, while the number of buses is $b = \int_{i\in[0,\mathcal{I}]} \frac{1}{B} v_b^i \, di$.

Commuter's features

Every commuter has to make a single modal choice, subject to her own individual features.

Assumption 5 Endowments. *Commuters have some monetary and time exogenous endowments, as well as they all are one-seater car owners:*

5.1. Each commuter $i \in \mathcal{I}$ is endowed with T units of time at the slot just before the working schedule, which can only be devoted to traveling or leisure activities (e.g., sleep more and wake up later).

5.2. Commuters are heterogeneous in their (per day) monetary wages, y^i.[5]

5.3. Each consumer is a one-seater car owner, all about the same brand.[6]

Assumption 6 Preferences. *Each commuter i's welfare is enhanced by*

- *increasing the consumption of a good inelastically supplied by the productive sector, c^i;*
- *increasing the leisure time not devoted to traveling at the peak-hour slot, l^i; and,*
- *reducing an idiosyncratic bad θ^i concerning traveling, whose input variables are the following:*

  i) travel time, both by car or bus, i.e., t_m^i, with $m = c$ or b;

[5]We must note that in this model the labor income of the commuter is fixed, as the working schedule was set at a previous decision stage.

[6]In practice, vehicle occupancy for vehicle for trips is very low, about 1.1 (see Parry, 2002, ft. 7).

ii) travel uncomfortably because of taking a particular transportation mode, i.e., β^i_m with $m = c$ or b, where β^i_c/β^i_b is the commuter i's exogenous mode disliked ratio between car and bus; and,

iii) the commuter i's valuation of the trip time, a heterogeneous variable with a distribution function $f(i)$ on the set of commuters $\mathcal{I}$.

We will assume that all commuters have identical disliked parameters, $\beta^i_m = \beta_m$ for all $i \in \mathcal{I}$ and $m = c, b$; and that bus trips are more uncomfortable than car trips, $\beta_c < \beta_b$. Besides, we will assume a uniform distribution function $f(i)$ for the heterogeneous valuation of trip time, so $f(i) = 1/\mathcal{I}$ for $i \in \mathcal{I}$. Then, the idiosyncratic production function of this bad for each commuter $i \in \mathcal{I}$ will be assumed to be

$$\theta^i = \theta(t^i_c, t^i_b, \beta^i_c, \beta^i_b, i) = [\beta_c v^i_c t^i_c + \beta_b v^i_b t^i_b] i/\mathcal{I}.$$

Finally, we will assume that commuter i's preferences can be represented by the continuous utility function

$$U^i\left(c^i, l^i, \theta^i\right) = c^i + \phi l^i - \frac{i}{\mathcal{I}}\left[\beta_c v^i_c t^i_c + \beta_b v^i_b t^i_b\right], \tag{1}$$

and we will assume that travel time is more valuable than leisure time, i.e., $\beta_m > \phi$ for $m = c, b$.

Observe that consumption and leisure are not perfectly substitutes in (1). The reason is that we cannot reduce leisure to increase labor –and, then, resources–, so that consumption can be risen. That is, the opportunity cost of leisure at trip-to-work peak hour is not labor, since leisure activities are developed off the working schedule.[7]

Finally, the treatment of the transportation services as a bad in our model deserves some comments. First, Assumption 3 stated the travel requirement to undertake working activities, so transportation services are an intermediate input in order to earn income. Accordingly, traveling *per se* does not enhance commuter's welfare, in contrast to the conventional transportation and congestion models (see Arnott, 2001).[8] Second, the time spent traveling might be painful for travelers, because its uncomfortable features, or pleasant as other activities can be undertaken simultaneously (see Oort, 1969, or Johnson, 1966). As long as only working trips are considered, no consideration is made on the case where commuters could derive

[7]Train and McFadden (1978), for example, mistakenly consider so.

[8]Such an assumption found in several papers is often made to create a demand for transportation, as welfare services provided by transportation represents the welfare of available goods consumed at destiny after traveling. However, this representation could also result in ridiculous consequences such as consumer's spare time and resources will be devoted, for example, to travel by bus instead of undertaking other leisure or consumption activities, because traveling by bus increases *per se* consumer's welfare.

certain welfare on the time they travel, e.g., sightseeing, read a book or a newspaper. Consequently, commuters disagree with spending time traveling to work at some transportation mode.

3.2 The production of the travel time

Each commuter takes a modal choice to consume the commodity *transportation along the link A to B at a particular slot of time.* This decision relies on the amount of resources wasted traveling, and Assumption 2 shows that the key variable for determining these costs is summarized in *travel time.* This variable depends on three elements: i) the features of the infrastructure; ii) the characteristics of the transportation modal system; and, iii) the amount of vehicles on road. While the first two are technological and were described at Assumptions 1 and 2 respectively; the latter falls into the realm of the economic theory. Travel time plays a crucial role in this paper, so it is worth to study it in detail.

3.2.1 Implications of technological features on travel time: "traffic flow" and "free-flow travel time"

Initially, we will focus on the technological elements affecting travel time by defining traffic flow, and then we will determine the average travel time for each mode.

Definition 4 *Let be I the number of commuters traveling at the peak-hour. Assume that c of them travel by car, and the remaining $I - c$ by bus.* **Traffic flow in the link A to B at a particular slot of time** *is the total number of vehicles simultaneously using the network. It will be denoted by $V = c + b$, that is, the total number of individual cars plus the total number of buses $b = (I - c)/B$, where B is the passengers capacity of a bus (Assumption 2.1).*

Observe that the traffic stock is bounded: $V \in [I/B, I]$, where I/B is the traffic flow in the case all agents travel by bus, and the upper bound is the case where all agents travel on their own car. Next, we define the set of all feasible combinations of cars and buses.

Definition 5 *Let be I the number of commuters traveling at the peak-hour. The* **traffic flow line** *is the set of the feasible pairs cars-buses (c, b) that are able to transport all commuters simultaneously,*[9]

$$\mathcal{V} = \{(c, b) \ s.t. \ I = c + Bb, \ with \ c \geq 0 \ and \ b \geq 0\} . \tag{2}$$

[9] Observe that Assumption 5.3 implies that all commuters have to take a modal decision as they all are car owners. However, if we would consider additionally commuters that do not take any modal decision, as they have no cars, they will have to travel by bus. Then the traffic flow line would move upwards by increasing an exogenous number of $\underline{b}$ buses: $\mathcal{I} = c + B(b + \underline{b})$.

The concept of traffic flow is an "accounting" definition of the commuters distribution, as no infrastructure or modal features play a role. Next, we consider such technological characteristics to consider a particular case of traffic flow.

Definition 6 *Consider Assumptions 1–2 are verified, i.e., the technological features of the road infrastructure and the transportation modes.* **Traffic free-flow in the link A to B at a particular slot of time** *is the case where the technological combination between travel time and transportation modes is not affected by the number of vehicles using the link.*

Two straightforward consequences of the definition of traffic free-flow emerges. Firstly, there is a threshold $\overline{V}$ on the traffic flow below of which the number of vehicles using simultaneously the link produces no mutual interaction among them. This is a technological value which depends on the transportation mode characteristics, as well as the road network features, Assumptions 1–2. The second consequence refers to the determination of a **free-flow travel time** for each of the modes: whenever there is no mutual interaction among vehicles on road a transportation mode m will last τ_m units of time to cover the distance from A to B, with $m = c$, b. It seems reasonable to think that under no mutual interaction car drivers arrive first, $\tau_c < \tau_b$. Finally note that both free-flow travel times are exogenous parameters that exclusively depend on the technological features, and are independent of the commuters' decisions.

3.2.2 Implications of economic decisions on travel time: "traffic congestion"

Yet, mutual interaction among vehicles could exist, so commuters trip might last for longer than the free-flow travel time. Consequently, travel time spent traveling at some mode m is the output of a technology that makes use as inputs the technological infrastructure and modal features, as well as the traffic flow considered as the result of all commuters simultaneous individual decisions. Whenever no vehicle interaction exists, i.e., $V \leq \overline{V}$, a trip will take its free-flow travel time, $t^i_m = \tau_m$, with $m = c$, b. If the traffic flow increases beyond the threshold $\overline{V}$ the presence of an extra vehicle slows down the average speed of all other drivers and rise their travel time, so a negative externality arises, denoted to by "*traffic congestion.*" Accordingly, the delay on commuters' travel time due to traffic congestion, with respects to the no-mutual interaction benchmark case, will be the way to measure the externality. Next, we define this negative externality (See Mas-Colell *et al*, 1995, p.352)

Definition 7 Traffic congestion *(or* **"congestion"***)* **in the link A to B at a period of time** *is present whenever the well-being of every commuter is directly affected by the travel decision of any other commuter. These simultaneous decisions disturb their technology combination between travel time and the modal choice with* *respect to the traffic free-flow case, because of the number of vehicles in the link.*[10]

3.3 The technology to produce travel time

We will now focus on the technology that produces travel time. Given that free-flow travel time τ_m gathers the exogenous infrastructure and modal features, and that traffic flow affects travel time beyond certain threshold, we may represent this technology as a function of these two inputs. We will assume that the congestion technology affects the travel time multiplicatively, as usually done in the literature (e.g., Parry 2002 or the US Bureau of Public Roads –see Anderson and Mohring 1996):

$$t_m^i = z_m(\tau_m, V) \equiv \tau_m(1 + \gamma(V)) \text{ with } m = c, b,$$

where $\gamma(V)$ represents the *production function of the congestion externality*, that is, the negative effect of traffic congestion on travel time. This function takes value zero for those levels of traffic flow below or equal to the threshold $\overline{V}$, and it is an increasing and convex function beyond, i.e. $\gamma'(V) > 0$ and $\gamma''(V) \geq 0$ when $V \geq \overline{V}$. Our analysis will follow at the general case. However, several parametrizations can be assumed. For example the linear functional form

$$\gamma(V) = \begin{cases} 0 & \text{if } V \leq \overline{V}; \\ \gamma \ \ [V - \overline{V}] & \text{if } V \geq \overline{V}; \end{cases} ;$$

or the one made use by the US Bureau of Public Roads (see Anderson and Mohring 1996)

$$\gamma(V) = \begin{cases} 0 & \text{if } V \leq \overline{V}; \\ \gamma \ \ (V/\overline{V})^4 & \text{if } V \geq \overline{V}, \end{cases} ,$$

where $\overline{V}$ is the capacity of the link (with $\gamma = 0,15$).

3.4 Traffic congestion: a first appraisal from welfare economics

The study of traffic congestion presents two particular features that distinguishes it from the standard externality literature. Firstly, traffic congestion does not directly affect commuters' welfare despite of being a negative externality, as in other

[10] Our definition is in tune with others outside economic grounds. For example, the workable definition of traffic congestion provided by Lomax, Turner, Shunk *et al* (1997) who defined two terms: "Congestion is travel time or delay in excess of that normally incurred under light or free-flow travel conditions; and, unacceptable congestion is travel time or delay in excess of an agreed-upon norm. The agreed-upon norm may vary by type of transportation facility, travel mode, geographic location and time of day." Of course, "agreed-upon norm" falls into what economists call *preferences*. Other similar definition is "congestion is the impedance vehicles impose on each other, due to the speed-flow relationship, in conditions where the use of a transport system approaches its capacity."ECMT (1999)

examples referred to in the public economics literature, such as pollution. Congestion affects commuters indirectly, since their welfare decreases the longer each trip takes: first, by increasing the unpleasant time traveling at some of the transportation modes; second, by lowering time devoted to leisure activities; and finally, only for car users, the monetary costs of car increases due a higher travel time, reducing consumption.

A second difference refers to traffic congestion does not affect the commuters' welfare until the aggregate variable *traffic flow* exceeds a certain exogenous technological bound $\overline{V}$. Such a threshold depends on the mode and network features (Assumptions 1–2), and defines a technological combination of cars and buses on the traffic flow line that generates the congestion threshold. This allows us to define the following set,

Definition 8 *Let be $\overline{V}$ a threshold on the traffic flow, which depends on the mode and network features (Assumptions 1–2), below of which the number of vehicles using simultaneously the link produces no mutual interaction among them. We define the* **congestion threshold set** *as the pair of cars and buses (c, b) for which traffic flow exactly coincides with the threshold capacity $\overline{V}$:*

$$\Phi(c, b) = \{(c, b) \ s.t. \ \overline{V} = \varphi(c, b), \ with \ c, \ b \geq 0\}. \tag{3}$$

[Hipotese: $b = \widetilde{\varphi}(c)$ is monotonically increasing, $\widetilde{\varphi}'(c) > 0$, to guarantee uniqueness.]

We will be assumed a linear functional technology[11]

$$\varphi(c, b) \equiv c + \varphi b,$$

with $\varphi > 0$.[12] This set separates the car-bus space into two regions where time delays exists or not. The upper contour set of Φ represents the pairs cars-buses such that the corresponding traffic flow entails a travel time delay; in contrast, the lower contour set of Φ shows up car-bus combinations with traffic free-flow.

A joint analysis of the traffic flow line (2) and the congestion threshold set (3) depicts several scenarios concerning traffic congestion for a given network and modal features. Considering the linear case, it will be expected that the slope of the traffic flow line is higher than the congestion threshold line, i.e. $1/\varphi > 1/B$ that implies $B > \varphi$.[13] Then, three cases are possible:

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[11] Observe that if we would consider additionally bus commuters that do not take any modal decision, as they have no cars the function φ will move upwards; that is, $\overline{V} = c + \varphi(b + \underline{b})$. However, none combination would be placed at the right of $(n, \underline{b})$.

[12] A bus requires roughly 3 passenger car units, but actually carries well over three times as many passengers as a passenger car (see Road Research Laboratory, 1965, p.200-201).

[13] The number of cars per bus to take out from roads to transport the same number of individuals must be higher than the number of cars per bus to keep the road without congestion within the congestion threshold set. Empirical studies show that $\varphi = 1,5$ while bus capacity exceeds this value, e.g., 20 passengers.

Case 1) Congestion will never exist whenever the infrastructure would allow all commuters to travel by car without any mutual vehicle interaction, i.e., $\overline{V} \geq \mathcal{I}$; that is, when the traffic flow line (2) falls below the congestion threshold set (3).

Case 2) There is always congestion in the road network whenever travel time delay exists even though there are only bus commuters, i.e., $\overline{V} \leq \varphi\mathcal{I}/B$; that is, whenever the congestion threshold set (3) falls below the traffic flow line (2).

Case 3) A less trivial outcome is the one where both sets intersect, the case of great interest for transportation economics and policy (see Figure 1). The crossing of (2) and (3) determines the *non-congestion threshold car-bus pair* $(\overline{c}, \overline{b})$ where all commuters travel and no congestion exists marginally. Such a modal threshold allocation separates the traffic line into two in terms of the existence of the externality:

(a) There will be a non-congested network whenever the modal distribution of commuters entails less cars and more buses on roads than in the threshold pair, i.e. $c < \overline{c}$ and $b > \overline{b}$, as any of such pairs $(c, b) \in \mathcal{V}$ falls below the restriction (3); and,

(b) There exists traffic congestion, and then travel time delays, because there are more cars and less buses on roads than in the threshold pair, i.e. $c > \overline{c}$ and $b < \overline{b}$. In such a case the pairs $(c, b) \in \mathcal{V}$ are placed above (3), a situation known as *network undercapacity.*

[Insert Figure 1]

To conclude, observe that despite we have not still computed centralized and decentralized mechanism allocations in our Walrasian equilibrium transportation model, the approach followed in this paper allows us to derive some consequences from Figure 1 in terms of the First Theorem of Welfare Economics. First, a decentralized modal distribution will be efficient at the non-congested region (Cases 1 and 3.a) as long as no externality exists, and no transportation policy is required. Instead, whenever travel time is affected by mutual vehicle interaction, the decentralized modal decisions that result in commuters allocation between cars and buses will not be Pareto optimal (Cases 2 and 3.b). For this case, the traffic congestion externality exists, as well as a social optimum level of congestion. Given the informational problems to achieve this first-best solution, policymakers would be interested in implementing second-best transportation policies to recover some efficiency. Which of these policies is the appropriated one, requires to find previously the social planner and decentralized outcomes, as done in the following sections.

4 The Pareto efficient problem, the indifferent commuter and the optimal congestion

In this section we characterize the Pareto optimal modal allocations in our environment and, consequently, the optimal level of traffic congestion. To this goal, we study the social planner problem for any given set of weights $\{\alpha_i\}_{i\in\mathcal{I}}$, posed in the Appendix A.1, to obtain the first order conditions (A1.1)-(A1.7) there stated. The optimal modal choice condition for each commuter is found as follows. First, substitute (A1.1)-(A1.2) onto (A1.3)-(A1.6), and then substitute the Lagrangian multiplier ν_2^i so as to equalize (A1.5) and (A1.6). Then, after multiplying by $v_c^i v_b^i$, we attain

$$v_c^i v_b^i \left\{ \left[\alpha_i \left(k_c t_c^i + \phi t_c^i + \frac{i}{\mathcal{I}} \beta_c t_c^i \right) + \psi \right] - \left[\alpha_i \left(k_b + \phi t_b^i + \frac{i}{\mathcal{I}} \beta_b t_b^i \right) + \frac{1}{B} \psi \right] \right\} = 0. \tag{4}$$

Some agents will be car commuters, i.e., $v_b^i = 0$, while others will be bus commuters, i.e., $v_c^i = 0$. At the margin, the optimal modal condition for the *optimal indifferent commuter* denoted to as $\hat{\imath}$, i.e., the one who is indifferent between traveling by car or by bus, is given by

$$\alpha_{\hat{\imath}} \left(k_c t_c^{\hat{\imath}} + \phi t_c^{\hat{\imath}} + \frac{\hat{\imath}}{\mathcal{I}} \beta_c t_c^{\hat{\imath}} \right) + \psi \;=\; \alpha_{\hat{\imath}} \left(k_b + \phi t_b^{\hat{\imath}} + \frac{\hat{\imath}}{\mathcal{I}} \beta_b t_b^{\hat{\imath}} \right) + \frac{1}{B}\psi, \tag{5}$$

where the Lagrangian multiplier for the congestion constraint is found by substituting (A1.3) and (A1.4) onto (A1.7)

$$\psi \;=\; \frac{\gamma'(V)}{1+\gamma(V)} \int_{i\in\mathcal{I}} \alpha_i \left[v_c^i t_c^i \left(\frac{i}{\mathcal{I}} \beta_c + k_c + \phi \right) + v_b^i t_b^i \left(\frac{i}{\mathcal{I}} \beta_b + \phi \right) \right] di. \tag{6}$$

It will be useful to introduce some notation. The optimal condition (4) allow us to define the auxiliary function on the commuter type as the difference of two increasing linear functions on the commuter type, which each represent a microeconomic justification of the generalized cost of transportation,

$$\begin{aligned} \Delta(i,\gamma,\psi) &= MC_b(i,\gamma,\psi) - MC_c(i,\gamma,\psi) \qquad (7)\\ MC_c(i,\gamma,\psi) &= \alpha_i \left(k_c t_c^i + \phi t_c^i + \frac{i}{\mathcal{I}} \beta_c t_c^i \right) + \psi \\ MC_b(i,\gamma,\psi) &= \alpha_i \left(k_b + \phi t_b^i + \frac{i}{\mathcal{I}} \beta_b t_b^i \right) + \frac{1}{B}\psi. \end{aligned}$$

The second function represents the generalized price of traveling by car for each type of commuter, that is, her marginal cost of traveling by car $[MC_c]$, while the latter represents the generalized price of traveling by bus for each type of commuter, that is, her marginal cost of traveling by bus $[MC_b]$. These generalized prices are compounded of four analogous costs. Three of them are private costs: an objective travel cost that decreases the monetary resources for purchasing additional consumption goods, $k_c t_c^i$ and k_b respectively; a subjective monetary valuation of travel time that reduces the commuter i's temporal resources for devoting to leisure activities, ϕt_m^i with $m = c$, b; and, the subjective monetary value of the commuter i's idiosyncratic welfare costs because of traveling by car or by bus, i.e., $\frac{i}{\mathcal{I}}\beta_m t_m^i$ with $m = c$, b. Finally, there exists an additional aggregate social cost that internalizes the congestion externality on the decision of traveling by car, ψ. Such a multiplier found in (6) is in fact a function of the traffic flow, i.e., $\psi = \psi(V)$. It will be active provided the congestion externality exists, and it is positive because congestion has a negative effect on each commuter's welfare costs increase. This is the case for any traffic flow beyond the technical traffic flow threshold $\overline{V}$, an exogenous parameter that depends on the road network and the transportation modal features (Assumptions 1–2).

The social planner's optimal rule for commuter modal allocation is the following. For a given traffic flow V, the commuter i' will travel by car whenever her generalized price of traveling by bus is dearer than that when traveling by car, $MC_b(i', \gamma(V), \psi(V)) > MC_c(i', \gamma(V), \psi(V))$, i.e., $\Delta(i', \gamma(V), \psi(V)) > 0$; otherwise, she will be a bus commuter since traveling by car is more expensive for her, $MC_b(i', \gamma(V), \psi(V)) < MC_c(i', \gamma(V), \psi(V))$, i.e., $\Delta(i', \gamma(V), \psi(V)) < 0$. At the margin, condition (5) allows us to characterize the *optimal indifferent commuter* $\hat{i}$, for which both generalized prices equalize; that is, $\hat{i}$ is a root of the auxiliary function $\Delta(i, \gamma(V(i)), \psi(V(i)))$, where the traffic flow is given by

$$V(i) = \int_0^i \frac{1}{B} di + \int_i^{\mathcal{I}} di = \frac{i}{B} + [\mathcal{I} - i]. \tag{8}$$

Observe that, implicitly, we are assuming that there exists at most only one marginal commuter that separates car-commuters from bus-commuters. Hence, we will make use an abuse of notation by denoting $\psi(V(\hat{i}))$ and $\gamma(V(\hat{i}))$ simply by $\psi(\hat{i})$ and $\gamma(\hat{i})$, respectively.

Next, we will study the solution of the problem considering two cases where there exists a congestion externality or not.

4.1 The case without traffic congestion.

Let us initially consider that the social planner allocates the commuters in such a way that traffic flow does not generate traffic congestion, i.e., $V(\hat{i}) < \overline{V}$. In this case, there are no external effect among commuters' modal allocation, so that the multiplier ψ is zero at (6), provided $\gamma(V) = 0$ and $\gamma'(V) = 0$. For such a case the travel time is the one under free-flow $t_m^i = \tau_m$ for $m = c$, b. Consequently, we will focus on the car-bus allocations (c, b) at the subset of the traffic flow line (2) below the congestion threshold set (3), i.e., $c \in [0, \overline{c}]$ and $b \in [\overline{b}, \mathcal{I}/B]$ (see Figure 1).

In this region it is easy to characterize the indifferent commuter $\hat{i} \in [B\overline{b}, \mathcal{I}]$ from condition (5). The auxiliary function on the commuter type (7), turns to be linear

$$\Delta(i,0,0) = k_b + \phi\tau_b - (k_c - \phi)\tau_c + [\beta_b\tau_b - \beta_c\tau_c]\frac{i}{\mathcal{I}} = \Delta(0,0,0) + \Delta'(0)i,$$

whose intercept is the value of the function for the commuter type $i = 0$ that only pays attention to monetary and temporal costs, i.e., $\Delta(0,0,0) = MC_b(0,0,0) - MC_c(0,0,0) = \phi(\tau_b - \tau_c) - (k_c\tau_c - k_b)$; and, it has positive slope $\Delta'(0) = [\beta_b\tau_b - \beta_c\tau_c]/\mathcal{I} > 0$, owing to the subjective valuation of the uncomfortable travel time is higher for bus trips than for car ones for each type of commuter, and that cars go faster and are more comfortable, i.e., $\beta_b > \beta_c$ and $\tau_b > \tau_c$ respectively.

The optimal indifferent commuter for the case without traffic congestion.

There will exist an indifferent commuter at the non-congested region whenever the monetary savings traveling by bus are able to offset the monetary value of the longer travel time by bus, that is, whether $\Delta(0,0,0) < 0$; otherwise, all commuters travel by car, i.e., $V = \mathcal{I} < \overline{V}$. Observe again that the indifferent commuter $\hat{i} \in [B\overline{b}, \mathcal{I}]$ is the root of the auxiliary function $\Delta(i,0,0)$, and is attained by

$$\hat{i}(\tau_c, \tau_b, k_c, k_b; 0) = \frac{\tau_c k_c - k_b + \phi(\tau_c - \tau_b)}{\beta_b\tau_b - \beta_c\tau_c}\mathcal{I} = \frac{\Delta(0,0,0)}{\Delta'(0)}.$$

[Insert Figure 2]

See Figure 2. The commuters are allocated as follows: there will be $\hat{i}$ commuters of type $[0, \hat{i}]$ that travel by bus, i.e., $v_b^i = 1$, so that the optimal number of buses are given by $\widehat{b} = \hat{i}/B$; while $\widehat{c} = \mathcal{I} - \hat{i}$ commuters of type $[\hat{i}, \mathcal{I}]$ will travel by car, i.e., $v_c^i = 1$. That is, commuters that disagree more the uncomfortable trips will travel by car, and those who disagree less will travel by bus. Consequently, the traffic flow (8) is given by

$$V(\hat{i}) = \frac{\hat{i}}{B} + [\mathcal{I} - \hat{i}] = \mathcal{I} - \frac{B-1}{B}\frac{\Delta(0,0,0)}{\Delta'(0)} < \overline{V}.$$

We summarize the modal allocation outcome for the case without traffic congestion in the following proposition.

Proposition 1 *Let us consider a network defined by Assumptions 1-2 that defines a congestion threshold $\overline{V} = V(\bar{i})$, and the modes features and commuters characterized by Assumptions 3-6. Let us consider that the traffic flow does not generate a congestion externality, i.e., $V(\hat{i}) < \overline{V}$, as the indifferent commuter is set at $\hat{i} \in [\bar{i}, \mathcal{I}]$. Then,*

i) If $\Delta(0,0,0) > 0$ all commuters will travel by car $(\hat{c}, \hat{b}) = (\mathcal{I}, 0)$, and the indifferent commuter is set at $\hat{i} = 0$ and the traffic flow is $V(0) = \mathcal{I} < \overline{V}$.

ii) If $\Delta(0,0,0) < 0$ *then*

ii.1) if $\Delta(0,0,0)/\Delta'(0) < \mathcal{I}$, *then* $\hat{i} = \Delta(0,0,0)/\Delta'(0)$ *and* $(\hat{c},\hat{b}) = (\mathcal{I}-\hat{i}, \hat{i}/B)$, *and the traffic flow* $V(\hat{i}) < \overline{V}$ *is found at (8).*

ii.2) if $\Delta(0,0,0)/\Delta'(0) > \mathcal{I}$ *all commuters will travel by bus,* $(\hat{c},\hat{b}) = (0,\mathcal{I}/B)$ *and then* $\hat{i} = \mathcal{I}$ *and the traffic flow is* $V(\mathcal{I}) = \mathcal{I}/B < \overline{V}$.□

Finally, the *non-congestion threshold car-bus pair* $(\overline{c},\overline{b})$ at the crossing of (2) and (3) sets a bound for the indifferent commuter that yield no external effects. (See Figure 2.) Given the road network and the transportation modal features (Assumptions 1–2), represented by the technological congestion threshold $\overline{V}$, we can define the *indifferent commuter congestion threshold* as the minimum number of commuters $\overline{i}$ that have to travel by bus in order that congestion does not exist. Hence, substituting $(\overline{c},\overline{b}) = (\mathcal{I}-\overline{i}, \overline{i}/B)$ into (3), we find $\overline{i}(B,\overline{V}) = B[\mathcal{I}-\overline{V}]/[B-\varphi]$. Accordingly, provided $\hat{i} \geq \overline{i}$ the optimal solution corresponds to the case where the optimal modal allocation results in a no-congestion externality; otherwise, the case of study is the congested one. Then we find a parametric lower threshold on the set of technological parameters that produces no external effects,

$$\frac{\Delta(0,0,0)}{\Delta'(0)} = \frac{\tau_c k_c - k_b + \phi(\tau_c - \tau_b)}{\beta_b\tau_b - \beta_c\tau_c}\mathcal{I} \geq B\frac{\mathcal{I}-\overline{V}}{B-\varphi}.$$

4.2 The case with traffic congestion.

Assume now that the social planner allocates the commuters in such a way that the indifferent commuter $\hat{i}$ defines a traffic flow that generates traffic congestion. Following the above reasoning, the indifferent commuter type is $\hat{i} < \overline{i}$, and the traffic flow resulted in (8) implies that $V(\hat{i}) > \overline{V}$, so $\gamma(V(\hat{i})) = \gamma V(\hat{i}) > 0$ and then $t_m^i = \tau_m(1+\gamma V(\hat{i}))$ for $m = c$, b. Consequently, we will focus on the car-bus allocations (c,b) at the subset of the traffic flow line (2) above the congestion threshold set (3), i.e., $c \in [\overline{c}, \mathcal{I}/B]$ and $b \in [0,\overline{b}]$ (see Figure 1). In this case there exists external effects on any commuter modal decision, i.e., $\psi > 0$ in (6), because $\gamma'(V)(\hat{i}) > 0$.

The social planner will choose the indifferent commuter $\hat{i} \in [0,\overline{i}]$, and will allocate commuters between travel modes by making use the auxiliary function on the commuter type (7)

$$\Delta(i,\gamma(\hat{i}),\psi(\hat{i})) = [k_b + \phi\tau_b - (k_c-\phi)\tau_c](1+\gamma(\hat{i})) - k_b\gamma(\hat{i}) + [\beta_b\tau_b - \beta_c\tau_c](1+\gamma(\hat{i}))\frac{i}{\mathcal{I}} + \frac{1-B}{B}\psi(\hat{i}),$$

where the effect of the externality found in (6) is a strictly positive value, provided $\hat{i} \geq 0$, given by

$$\psi(\hat{i}) = \gamma'(\hat{i})\left\{\frac{\Delta'(0)}{2}\hat{i}^2 + [\Delta(0,0,0) - k_b]\,\hat{i} + \mathcal{I}\left[\frac{\beta_c\tau_c}{2} + (k_c+\phi)\tau_c\right]\right\}. \quad (9)$$

Observe that in the case that the production function of the congestion externality is linear, i.e., $\gamma(\hat{\imath}) = \gamma V(\hat{\imath})$ for $\hat{\imath} \in [0, \bar{\imath})$, this external effect is a strictly positive convex function that achieves the maximum degree of congestion effect at $\hat{\imath}^* = [\Delta(0,0,0) - k_b]/\Delta'(0)$.

The indifferent commuter for the case with traffic congestion.

Owing to the indifferent commuter $\hat{\imath}$ is chosen by the planner, the auxiliary function (7) is also a linear function in the consumer type i, that is,

$$\Delta(i, \gamma(\hat{\imath}), \psi(\hat{\imath})) = \Delta(0, \gamma(\hat{\imath}), \psi(\hat{\imath})) + \Delta'(\gamma(\hat{\imath}))i, \tag{10}$$

whose slope is positive, $\Delta'(\gamma(\hat{\imath})) = (1+\gamma(\hat{\imath}))\Delta'(0) > 0$; and, the intercept is the value of the function for the commuter type $i = 0$ that only pays attention to monetary and temporal costs, i.e.,

$$\Delta(0, \gamma(\hat{\imath}), \psi(\hat{\imath})) = \Big[\Delta(0,0,0) + [\Delta(0,0,0) - k_b]\,\gamma(\hat{\imath}) + \frac{1-B}{B}\psi(\hat{\imath})\Big].$$

Observe that there will exists an indifferent commuter whenever this intercept is negative. For example, whenever the monetary savings traveling by bus are able to offset the monetary value of the longer travel time by bus, that is, whether $\Delta(0,0,0) < 0$. It is important to realize that the consideration of the congestion externality affects the modal allocation because its impact on the travel costs for the bus commuters are lower than on those for the car ones; accordingly, the net cost of the congestion externality is negative $(1-B)\psi/B < 0$. This means that the commuter may still travel by bus even if $\Delta(0,0,0)$ is positive, but her costs are not able to offset the negative congestion externality effect. Otherwise, if $\Delta(0, \gamma(\hat{\imath}), \psi(\hat{\imath})) > 0$ for all i, all commuters travel by car, that is, $\hat{\imath} = 0$, $(\hat{c}, \hat{b}) = (\mathcal{I}, 0)$, and $V = \mathcal{I} > \overline{V}$.

Next, we will characterize the indifferent commuter $\hat{\imath} \in [0, \bar{\imath}]$ for the congestion case as the root of the auxiliary function (10),

$$\delta(i) \equiv \Delta(i, \gamma(i), \psi(i)) = \Delta(0, \gamma(i), \psi(i)) + \Delta'(\gamma(i))i,$$

where the traffic flow is found at (8) and the external effect at (6). The following result presents the modal allocation for the congested case.

Proposition 2 *Let us consider a network defined by Assumptions 1-2 that defines a congestion threshold $\overline{V} = V(\bar{\imath})$, and the modes features and commuters characterized by Assumptions 3-6. Let us a assume any functional parametrization for $\gamma(V)$ and $\varphi(c, b)$. Let us consider that the traffic flow generates a congestion externality, i.e., $V(\hat{\imath}) > \overline{V}$, as the indifferent commuter is set at $\hat{\imath} \in [0, \bar{\imath})$. Then,*

i) If $\Delta(0, \gamma(\hat{\imath}), \psi(\hat{\imath})) > 0$ all commuters will travel by car $(\hat{c}, \hat{b}) = (\mathcal{I}, 0)$, and the indifferent commuter is set at $\hat{\imath} = 0$ and the traffic flow is $V(0) = \mathcal{I} > \overline{V}$.

ii) If $\Delta(0, \gamma(\hat{i}), \psi(\hat{i})) < 0$ *and* $\Delta(0, \gamma(0), \psi(0)) < 0$*, then,*

ii.1) if $\Delta(\bar{i}, \psi(\bar{i})) > 0$ *then there exists an indifferent commuter at* $\hat{i} \in [0, \bar{i})$ *for which* $\Delta(\hat{i}, \gamma(\hat{i}), \psi(\hat{i})) = 0$ *and such that* $(\hat{c}, \hat{b}) = (\mathcal{I} - \hat{i}, \hat{i}/B)$ *and the traffic flow* $V(\hat{i})$ *is found at (8).*

ii.2) if $\Delta(\bar{i}, \psi(\bar{i})) < 0$ *then* $\Delta(0, 0) < 0$ *and the function* $\delta(i)$ *is increasing for all* $i > 0$*, i.e.,* $\delta'(i) > 0$*. Then* $\hat{i} > \bar{i}$ *and this case drops into the non-congestion case at Proposition 1.ii).*

iii) If $\Delta(0, \gamma(\hat{i}), \psi(\hat{i})) < 0$ *and* $\Delta(0, \gamma(0), \psi(0)) > 0$*, then* $\Delta(0, 0, 0) > 0$ *and* $\Delta(\bar{i}, \psi(\bar{i})) > 0$*. Consequently, there exists at least one indifferent commuter at* $\hat{i} \in [0, \bar{i})$ *for which* $\Delta(\hat{i}, \gamma(\hat{i}), \psi(\hat{i})) = 0$ *and such that* $(\hat{c}, \hat{b}) = (\mathcal{I} - \hat{i}, \hat{i}/B)$ *and the traffic flow* $V(\hat{i})$ *is found at (8).*$\square$

Proof of Proposition 2. Parts i) and ii) are straightforward consequences of (10) and, for part ii) that $\delta(0) < \delta(\bar{i})$. To proof part iii) take the auxiliary function $\rho(i) \equiv \Delta(0, \gamma(i), \psi(i))$, such that $\rho(0) \equiv \Delta(0, \gamma(0), \psi(0)) > 0$; $\rho(\bar{i}) \equiv \Delta(0, \psi(\bar{i})) = \Delta(0, 0, 0) > 0$, with $\rho'(i) = 0$; and, $\rho(\hat{i}) \equiv \Delta(0, \gamma(\hat{i}), \psi(\hat{i})) = -\hat{i}\Delta'(1 + \gamma(\hat{i})))$. The latter is a consequence of $\Delta(i, \gamma(i), \psi(i)) = \Delta(0, \gamma(i), \psi(i)) + i(1 + \gamma(i))\Delta'(0)$.$\square$

5 The decentralized allocation of traffic.

In this section we will analyze the modal allocations when the modal decisions are not centralized, but taken instead by each each commuter independently. We will suppose that commuters behave stationary, so their modal choice is average. Accordingly, rather than focusing on some short-run (dynamic) adjustment process problem, for example, after the introduction of alternative roads or travel modes, we will analyze a (long run) stationary equilibrium, i.e. the static case once all commuters has been accommodated to the (daily) peak-hour transportation environment along the infrastructure. In this decentralized environment, commuters have property rights as depicted by Assumption 5. Each commuter i is a car owner; her wealth comes from an exogenous income y^i, e.g., her real wage per day; and, she is endowed with T units of time at the slot just before the working activities to devote traveling or leisure activities.

The commuter i's problem, for any $i \in [0, \mathcal{I}]$, consists on choosing the transportation mode to undertake the trip, whether by car or by bus, the leisure time at the peak-hour slot, and the consumption decision at off-peak hours that maximizes her welfare subject to her monetary and temporal restrictions. The problem and the first order conditions (A2.1)-(A2.6) are displayed in Appendix A.2.

The decentralized optimal modal choice condition for each commuter is found as follows. First, substitute (A2.1)-(A2.2) onto (A2.3)-(A2.6), and then substitute the

Lagrangian multiplier ν_2^i so as to equalize (A2.5) and (A2.6). Then, after multiplying by $v_c^i v_b^i$, we attain

$$v_c^i v_b^i \left\{ \left[k_c t_c^i + \phi t_c^i + \frac{i}{\mathcal{I}} \beta_c t_c^i \right] - \left[k_b + \phi t_b^i + \frac{i}{\mathcal{I}} \beta_b t_b^i \right] \right\} = 0.$$

Some agents will be car commuters, i.e., $v_b^i = 0$, while others will be bus commuters, i.e., $v_c^i = 0$. At the margin, the optimal modal condition for the *decentralized indifferent commuter* denoted to as i^*, i.e., the one who is indifferent between traveling by car or by bus, is given by

$$\left(k_c t_c^{i^*} + \phi t_c^{i^*} + \frac{i^*}{\mathcal{I}} \beta_c t_c^{i^*} \right) \quad = \quad \left(k_b + \phi t_b^{i^*} + \frac{i^*}{\mathcal{I}} \beta_b t_b^{i^*} \right). \tag{11}$$

It is worth of commenting that the commuter optimal decision is independent of the individual income, because of our assumption that the preferences (1) are quasilinear.

We make use the same notation as in the previous section so that the auxiliary function in (7) to determine the individual optimal decision as

$$\Delta(i, \gamma, 0) = MC_b(i, \gamma, 0) - MC_c(i, \gamma, 0).$$

It is important to realize that the commuters do not internalize their external costs at all, which means that their decisions are independent of the aggregate level of the traffic flow. Accordingly, the commuter's decisions are taken as no externality exists.

We can define an equilibrium as follows.

Definition 9 *Let us consider a network defined by Assumptions 1-2 that defines a congestion threshold* $\overline{V} = V(\bar{i})$*, and the modes features and commuters characterized by Assumptions 3-6. An* **equilibrium** *of this network is the consumption decision at off-peak hours, the leisure time at the peak-hour slot and the transportation mode to undertake the trip, whether by car or by bus, for each commuter* $i \in [0, \mathcal{I}]$*, i.e.,* $\{(c^{i*}, l^{i*}, v_c^{i*}, v_b^{i*})\}_{i \in [0,\mathcal{I}]}$*, and a traffic flow,* V^**, such that,*

(1) $(c^{i*}, l^{i*}, v_c^{i*}, v_b^{i*})$ *maximizes the commuter i's problem for a given traffic flow* V^**, and for any* $i \in [0, \mathcal{I}]$*;*

(2) Markets clears,

$$\begin{aligned} c^{i*} + \left[k_c \tau_c^i v_c^{i*} (1 + \gamma(V^*)) + k_b v_b^{i*} \right] &= y^i \text{ for each } i \in [0, \mathcal{I}] \\ l^{i*} + \left[v_c^{i*} \tau_c^{i*} + v_b^{i*} \tau_b^i \right] (1 + \gamma(V^*)) &= T \text{ for each } i \in [0, \mathcal{I}] \\ c^* + b^* = \int_{i \in [0,\mathcal{I}]} v_c^{i*} di + \frac{1}{B} \int_{i \in [0,\mathcal{I}]} v_b^{i*} di &= V^*. \end{aligned}$$

The decentralized optimal rule for modal allocation for each commuter is the following. For any given decentralized traffic flow $V^* = V(i^*)$ found at (8), the commuter i' will travel by car whenever her generalized price of traveling by bus is dearer than that when traveling by car, $MC_b(i', \gamma(i^*), 0) > MC_c(i', \gamma(i^*), 0)$, i.e., $\Delta(i', \gamma(i^*), 0) > 0$; otherwise, she will be a bus commuter since traveling by car is more expensive for her, $MC_b(i', \gamma(i^*), 0) < MC_c(i', \gamma(i^*), 0)$, i.e., $\Delta(i', \gamma(i^*), 0) < 0$.

At the margin, condition (11) allows us to characterize the *decentralized indifferent commuter* i^*, for which both generalized prices equalize; that is, i^* is a root of the linear auxiliary function

$$\Delta(i, \gamma(i^*), 0) = \Delta(0, \gamma(i^*), 0) + \Delta'(\gamma(i^*))i,$$

whose intercept is $\Delta(0, \gamma(i^*), 0) = MC_b(0, \gamma(i^*), 0) - MC_c(0, \gamma(i^*), 0) = \phi(\tau_b - \tau_c)(1 + \gamma(i^*)) - (k_c\tau_c(1 + \gamma(i^*)) - k_b)$; and, it has positive slope $\Delta'(\gamma(i^*)) = [\beta_b\tau_b - \beta_c\tau_c](1 + \gamma(i^*))/\mathcal{I} > 0$. Independently of being in the congested region or not, the indifferent commuter $i^* \in [0, \mathcal{I}]$ is attained by

$$i^*(\tau_c, \tau_b, k_c, k_b; \gamma(i^*)) = \frac{\tau_c k_c - k_b\frac{1}{1+\gamma(i^*)} + \phi(\tau_c - \tau_b)}{\beta_b\tau_b - \beta_c\tau_c}\mathcal{I} = \frac{\Delta(0, \gamma(i^*), 0)}{\Delta'(\gamma(i^*))}.$$

This function, somehow, could be considered as a demand function for the modes. Then, the decentralized equilibrium modal allocation is the following: there will be i^* commuters of type $[0, i^*]$ that travel by bus, and $\mathcal{I} - i^*$ commuters that travel by car. That is, commuters that disagree more the uncomfortable trips will travel by car, and those who disagree less will travel by bus, as each commuters demand for transportation modes are independent of their income. The equilibrium modal distribution is $(c^*, b^*) = (\mathcal{I} - i^*, i^*/B)$, so the traffic flow (8) is given by

$$V(i^*) = \frac{i^*}{B} + [\mathcal{I} - i^*] = \mathcal{I} - \frac{B-1}{B}\frac{\Delta(0, \gamma(i^*), 0)}{\Delta'(\gamma(i^*))}.$$

This analysis has been undertaken independently that there exists, or not, a congestion externality. The following proposition compares the equilibrium allocation with the optimal allocation found in the previous section for the case with and without externality.

Proposition 3 *Let us consider a network defined by Assumptions 1-2 that defines a congestion threshold $\overline{V} = V(\bar{i})$, and the modes features and commuters characterized by Assumptions 3-6. Let be i^* the decentralized indifferent commuter. Then,*

i) If the decentralized traffic flow is below the threshold capacity, i.e., $V(i^*) < \overline{V}$, then the decentralized indifferent commuter is the same as the optimal indifferent commuter, i.e., $i^* = \hat{i} < \bar{i}$, and the equilibrium allocation is Pareto efficient.

ii) Otherwise, if $V(i^*) > \overline{V}$, then the number of commuters traveling by car in equilibrium is higher than in the optimal case, so $\bar{i} < i^* < \hat{i}$, and the equilibrium allocation is not Pareto efficient.

iii) Finally, if $\hat{i} = 0$, then all commuters travel by car at the decentralized equilibrium $\bar{i} > i^* = \hat{i} = 0$, and the corner equilibrium allocation is Pareto efficient.

Proof of the Proposition 3. The proof is straightforward for *i*). In what respects to *ii*) recall that in the equilibrium case the commuters do not internalize their decisions on the aggregate traffic flow. Then, each commuter i's costs at the decentralized case will always be lower than in the optimal one, i.e., $MC_m(i, \gamma(i^*), 0) < MC_m(i, \gamma(i^*), \psi(i^*))$, for $m = b$, c. Provided the congestion externality hurts more to car commuters, if ψ are not considered more commuters will choose travel by car.□

Two comments on the descentralized allocation of traffic can be made. First, there exists a traffic threshold $\overline{V}$ such that below this threshold traffic level do not provoke congestion. In this case $\gamma(V) = 0$ and $\gamma'(V) = 0$ for all $V \leq \overline{V}$. In consequence $\psi = 0$ and the descentralized allocation is efficient. Beyond this value any level of traffic provoke congestion, so the resulting allocation is not Pareto optimum: the individual optimal conditions condition (A2.5) and (A2.6) are not equal to the social planner conditions (A1.5) and (A1.6). The reason is that drivers do not internalize their modal decision on other agents' travel time and on her own travel time. Self-interest maximization leads each agent to equate her private marginal rate (of substitution or transformation) to the price ratio and results in the equalization of private rates, whereas Pareto optimality requires the equalization of social rates.

[Insert Figure 2]

Second, the individual i's problem above described chooses only one transportation mode. Therefore, her problem can be reduced to[14]

$$\max\{\mathcal{U}_c^i, \mathcal{U}_b^i\} = \max\left\{U^i\Big(y^i - k_c(t_c^i), T - t_c^i, \theta^i(t_c^i, \beta_c, i)\Big); U^i\Big(y^i - k_b, T - t_b^i, \theta^i(t_b^i, \beta_b, i)\Big)\right\}.$$

Therefore, for each level of congestion, there exists a decentralized indifferent agent i^* such that $\mathcal{U}_c^{i^*} = \mathcal{U}_b^{i^*}$, and the equilibrium level of traffic flow, $V(i^*)$ is given by (8).

[14]The transportation literature usually starts from this expression. See Ben-Akiva and Lerman (1985, Chap.3). For example, p.45, taking an additive utility function $U^i(c^i, l^i, \theta^i) = \beta_1 c^i + \beta_2 l^i - \beta_3 \theta^i$. Then the problem would be reduced to

$$max\{\mathcal{U}_c^i, \mathcal{U}_b^i\} = \{\beta_{0c} - \beta_1 k_c(t_c^i) - \beta_2 t_c^i - \beta_3 \theta^i(t_c^i, \beta_c, i); \beta_{0b} - \beta_1 k_b - \beta_2 t_b^i - \beta_3 \theta^i(t_b^i, \beta_b, i)\}$$

where β_{0m} are the exogenous parameters for each mode $m = c$, b. This is the relations usually estimated in empirical works.

5.1 Welfare losses of traffic congestion.

Finally, the equilibrium framework we have made use along this paper allow us to gauge the welfare losses at the decentralized equilibrium at the congested case. Given a network defined by Assumptions 1-2 that defines a congestion threshold $\overline{V} = V(\bar{i})$, and the modes features and commuters characterized by Assumptions 3-6. Let be i^* the decentralized indifferent commuter, and $\hat{i}$ the optimal indifferent commuter. Then the welfare losses for each commuter i is given by comparing the welfare at the decentralized equilibrium from the one she would received at the optimal allocation. That is,

$$w^i(i^*, \hat{i}) = U^{i^*}\left(c^{i^*}, l^{i^*}, \theta^{i^*}\right) - U^{\hat{i}}\left(c^{\hat{i}}, l^{\hat{i}}, \theta^{\hat{i}}\right).$$

Recall that there are no welfare losses in the case that no congestion exists, because $i^* = \hat{i}$ and then the equilibrium modal distribution is Pareto efficient. At the aggregate level, the total aggregate welfare losses are given by the addition of all these individual losses, i.e.,

$$W^i = \int_{i \in [0,\mathcal{I}]} w^i di.$$

The computation of these welfare losses will help us to analyze any menu of second-best transportation policies for improving a decentralized network congestion.

6 Policies

The problem of congestion we are dealing has an interesting and particular feature that distinguishes this problem of externalities from others of the kind. Observe that the externality “congestion” does not exist until a certain degree of traffic –degree which depends on the road infrastructure. This is crucial when we are studying the Pareto-efficient problem. Below this threshold no externality exists, so the first welfare theorem points out that the decentralized problem is also one possible allocation found at the Social Planner problem. Beyond this level of traffic, congestion exists and then decentralized allocations are not efficient so government policies could be required.[15] However, given the existing infrastructures and the existing modes we could expect an efficient allocation of traffic supported with certain level of congestion.

The unique transportation policy for Case 2) congestion externality is an improvement on infrastructures or the introduction of a new mode.

7 Numerical example

We ilustrate the utility of this model obtaining the traffic allocation and congestion using american traffic data.

Tasks:

i. Determine indifferent conmmuter in centralised and decentralised equilibrium. So, determine traffic flow and congestion.

ii. Determine the welfare losses of nonoptimal allocations of traffic

iii. What does imply an inefficient modal allocation in cuantitative terms?

8 Conclusions

The goal of this paper has been to present a microeconomic foundated model of the traffic stream in a city at daily peak hours. We analyzed the social planner problem and characterized all Pareto optimal allocations. Then, we calculated the decentralized equilibrium. The main characteristic of congestion as an externality is that it is not always active. For low levels of traffic, no congestion exist and the decentralized allocations are efficient. However, for high levels of traffic, the externality appears, and then efficient and decentralized allocations are not the

[15]This aspect is important when the policy-maker has in mind to implement some policy, like an improvement of infrastructures. If the improvement is enough so that the externality disappears, an efficient allocation is obtained. That is, we must take care when studying policy recommendations in our static partial equilibrium model, since the money or resources to carry out infrastructures improvements are diverted from other alternative uses, and we should specify which.

We could assume that an improvement on the infrastructure -except for, e.g., an underground- may decrease citizens' welfare: think on streets with five lanes in two directions.

same. As in other examples of externalities, there may exists an optimal level of congestion.

Further research points out to analyze some kind of mechanisms that can be implemented to achieve the Pareto optimal allocations. Any of these mechanisms must include the extraction of rents from those agents who produces the negative externality, and a transfer of income to those who suffers it.[16]

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[16]This is a case of interest, since agents that moves from car to buses will be worsened and should be compensated. However, since their cars are taken out from streets, the congestion is reduced so they will also result improved.

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Appendix: The social planner Problem

A.1 The social planner problem

The social planner maximizes the commuters' weighted welfare function, by choosing an allocation of consumption, leisure, travel time at each of the modes, and the modal choice for each of the $\mathcal{I}$ commuters, which also entails the aggregate traffic flow. The planner is restricted by the feasible consumption set, where consumption and travel costs equals total income resources; and, each commuter's feasible time set, where time endowment is devoted to leisure activities and travel time, which is produced with aggregate modal decisions and the technological features described in Assumptions 1–2. Hence, the problem is the following,

$$\max_{\left\{c^i,l^i,t_c^i,t_b^i,v_c^i,v_b^i\right\}_{i\in\mathcal{I}},V} \int_{i\in[0,\mathcal{I}]} \alpha_i \left\{ c^i + \phi l^i - \tfrac{i}{\mathcal{I}}\left[\beta_c v_c^i t_c^i + \beta_b v_b^i t_b^i\right] \right\} di$$

$$\begin{array}{rrcll}
\text{s.t.} & \int_{i\in[0,\mathcal{I}]} \left[c^i + (k_c t_c^i v_c^i + k_b v_b^i) \right] di & = & Y \equiv \int_{i\in[0,\mathcal{I}]} y^i & : \lambda_M \\
& l^i + v_c^i t_c^i + v_b^i t_b^i & = & T \quad \text{for } i\in\mathcal{I} & : \lambda_T^i \\
& t_c^i & = & \tau_c\left[1+\gamma(V)\right] \quad \text{for } i\in\mathcal{I} & : \mu_c^i \\
& t_b^i & = & \tau_b\left[1+\gamma(V)\right] \quad \text{for } i\in\mathcal{I} & : \mu_b^i \\
& v_c^i v_b^i & = & 1 \quad \text{for } i\in\mathcal{I} & : \nu_1^i \\
& v_c^i + v_b^i & = & 0 \quad \text{for } i\in\mathcal{I} & : \nu_2^i;\ \text{and} \\
& \int_{i\in[0,\mathcal{I}]} \left[v_c^i + \frac{1}{B} v_b^i\right] di & = & V & : \psi
\end{array}$$

where $Y = \int_{i\in[0,\mathcal{I}]} y^i \; di$ is the aggregate consumption goods; α_i is the weighting assigned to commuter i by the planner; and, for a given set of the parameters τ_c, τ_b, β_c, β_b, k_c, k_b, ϕ, γ, $\overline{V}$, B, and $\mathcal{I}$.

The necessary conditions of the problem are given by the constraints at the planner problem multiplied by the corresponding Lagrangian multiplier, and the first order conditions are given by

$$\begin{array}{llllllllllll}
c^i & : & \alpha_i 1 & -\lambda_M & & & & & & & = 0 & \text{(A1.1)} \\
l^i & : & \alpha_i \phi & & -\lambda_T^i & & & & & & = 0 & \text{(A1.2)} \\
t_c^i & : & -\alpha_i \frac{i}{\mathcal{I}} \beta_c v_c^i & -\lambda_M v_c^i k_c & -\lambda_T^i v_c^i & -\mu_c^i & & & & & = 0 & \text{(A1.3)} \\
t_b^i & : & -\alpha_i \frac{i}{\mathcal{I}} \beta_b v_b^i & & -\lambda_T^i v_b^i & & -\mu_b^i & & & & = 0 & \text{(A1.4)} \\
v_c^i & : & -\alpha_i \frac{i}{\mathcal{I}} \beta_c t_c^i & -\lambda_M t_c^i k_c & -\lambda_T^i t_c^i & & & -v_b^i \nu_1^i & -\nu_2^i & -\psi & = 0 & \text{(A1.5)} \\
v_b^i & : & -\alpha_i \frac{i}{\mathcal{I}} \beta_b t_b^i & -\lambda_M k_b & -\lambda_T^i t_c^i & & & -v_c^i \nu_1^i & -\nu_2^i & -\frac{1}{B}\psi & = 0 & \text{(A1.6)}
\end{array}$$

for $i \in \mathcal{I}$, and

$$V \quad : \quad \gamma'(V) \int_{i\in[0,\mathcal{I}]} \left[\tau_c \mu_c^i + \tau_b \mu_b^i\right] di \quad +\psi \quad = \quad 0 \qquad \text{(A1.7)}$$

where λ_M, λ_T^i, μ_c^i, μ_b^i, ν_1^i, ν_2^i, and ψ are the Lagrangian multipliers, which are positive as long as individual preferences are monotonic.

A.2 The commuter i's problem

Any commuter i, with $i \in [0, \mathcal{I}]$, maximizes her utility $U^i(c^i, l^i, \theta(t_c^i, t_b^i))$ by choosing the transportation mode to undertake the trip, whether by car or by bus, the leisure time at the peak-hour slot, and the consumption decision at off-peak hours subject to her monetary and temporal restrictions, where time endowment is devoted to leisure activities and travel time, which is produced with aggregate modal decisions and the technological features described in Assumptions 1–2. Hence, the problem is the following,

$$\begin{array}{rcll}
\max_{\{c^i, l^i, t_c^i, t_b^i, v_c^i, v_b^i\}} & c^i + \phi l^i - \frac{i}{\mathcal{I}}\left[\beta_c v_c^i t_c^i + \beta_b v_b^i t_b^i\right] & & \\
\text{s.t.} \quad c^i + k_c t_c^i v_c^i + k_b v_b^i & = & y^i & : \lambda_M^i \\
l^i + v_c^i t_c^i + v_b^i t_b^i & = & T & : \lambda_T^i \\
t_c^i & = & \tau_c\left[1 + \gamma(V)\right] & : \mu_c^i \\
t_b^i & = & \tau_b\left[1 + \gamma(V)\right] & : \mu_b^i \\
v_c^i v_b^i & = & 1 & : \nu_1^i\text{; and} \\
v_c^i + v_b^i & = & 0 & : \nu_2^i,
\end{array}$$

given the exogenous income, y^i, and the traffic flow V; and, for a given set of the parameters τ_c, τ_b, β_c, β_b, k_c, k_b, ϕ, γ, $\overline{V}$, B, and $\mathcal{I}$. Observe that any commuter i takes the traffic flow as given, since each agent does not internalize her negative contribution to the increase of travel time when she makes her modal decision. The first order conditions are as follows:

$$\begin{array}{rclllllllcll}
c^i & : & 1 & -\lambda_M^i & & & & & & = & 0 & \text{(A2.1)} \\
l^i & : & \phi & & -\lambda_T^i & & & & & = & 0 & \text{(A2.2)} \\
t_c^i & : & -\frac{i}{\mathcal{I}}\beta_c v_c^i & -\lambda_M^i v_c^i k_c & -\lambda_T^i v_c^i & -\mu_c^i & & & & = & 0 & \text{(A2.3)} \\
t_b^i & : & -\frac{i}{\mathcal{I}}\beta_b v_b^i & & -\lambda_T^i v_b^i & & -\mu_b^i & & & = & 0 & \text{(A2.4)} \\
v_c^i & : & -\frac{i}{\mathcal{I}}\beta_c t_c^i & -\lambda_M t_c^i k_c & -\lambda_T^i t_c^i & & & -v_b^i \nu_1^i & -\nu_2^i & = & 0 & \text{(A2.5)} \\
v_b^i & : & -\frac{i}{\mathcal{I}}\beta_b t_b^i & -\lambda_M k_b & -\lambda_T^i t_c^i & & & -v_c^i \nu_1^i & -\nu_2^i & = & 0 & \text{(A2.6)}
\end{array}$$

where λ_M^i, λ_T^i, μ_c^i, μ_b^i, ν_1^i, and ν_2^i are the Lagrangian multipliers, which are positive as long as individual preferences are monotonic.

Figures

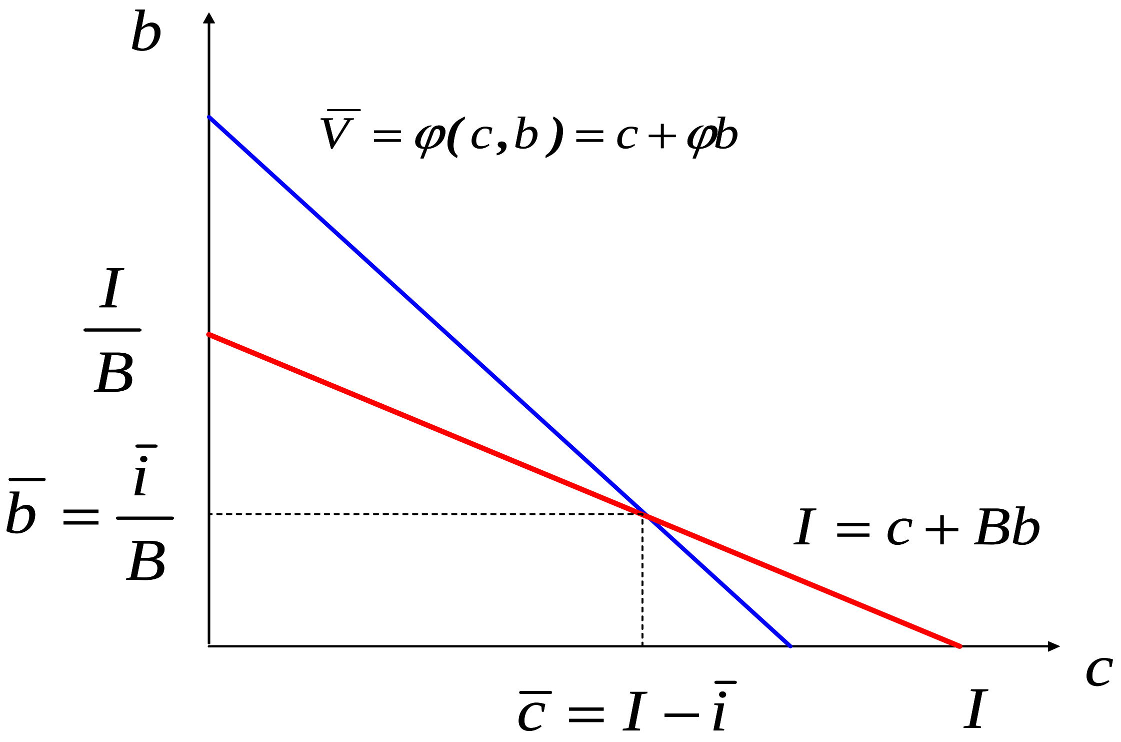

Figure 1: Traffic flow line (2), and the congestion threshold (3) set for a given technological congestion threshold $\overline{V}$. At the crossing the non-congestion threshold pair $(\bar{c}, \bar{b}) = (\mathcal{I} - \bar{i}, \bar{i}/B)$

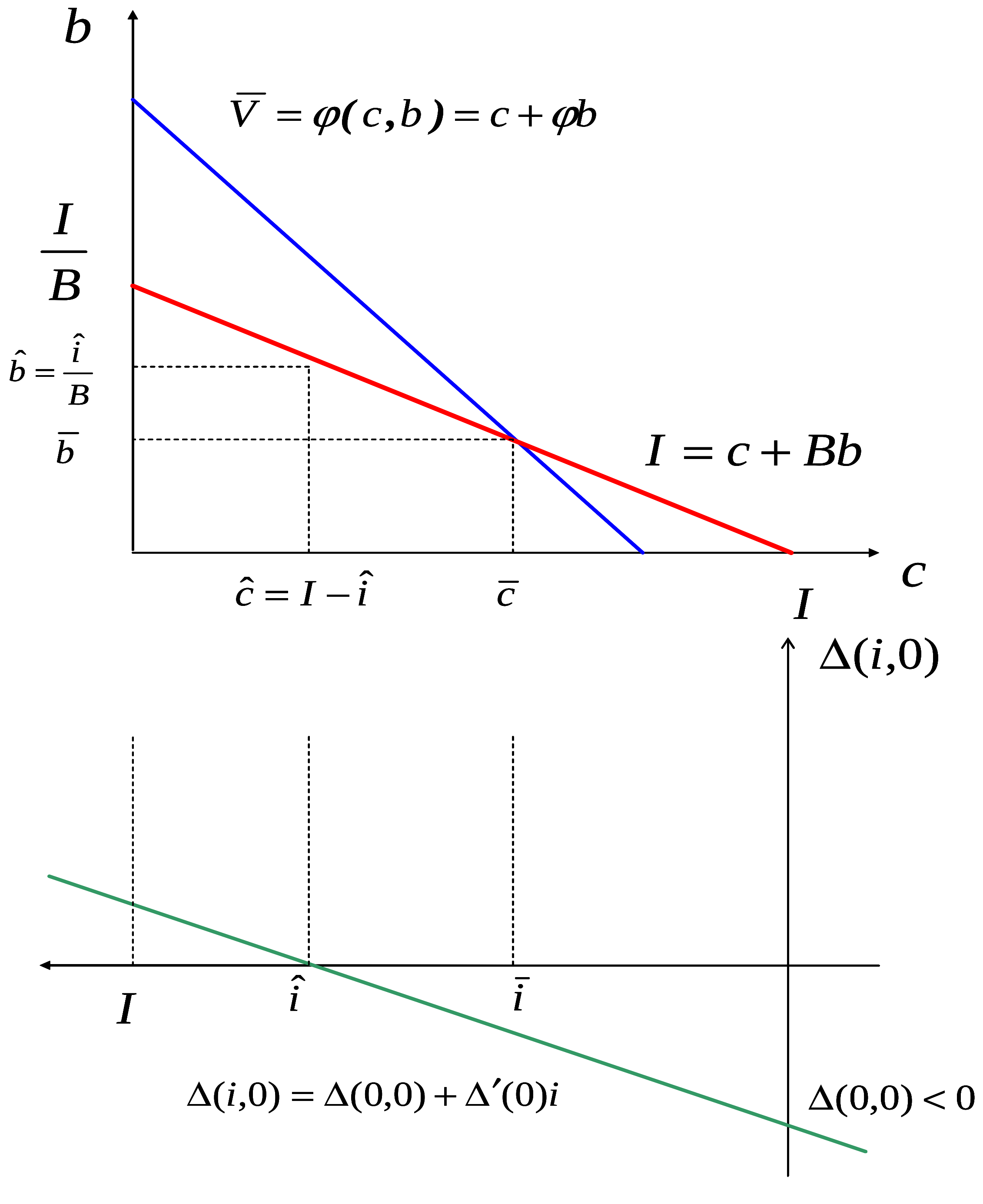

Figure 2: Optimal modal distribution $(\widehat{c}, \widehat{b})$ for the non-congested case. Above traffic flow line (2), and the congestion threshold (3) set for a given technological congestion threshold $\overline{V}$. Below, the auxiliary function (7) with positive axis at the left hand, for the case that the intercept takes a negative value $\Delta(0,0) > 0$.