Some Considerations on the Model of Endogenous Growth with Physical Capital, Human Capital and $R\&D$

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Abstract

The main aim of our paper is to try to improve the results obtained by Funke and Strulik in order to obtain optimal solutions for all variables. To do this we develop an alternative model of endogenous growth with physical capital, human capital and $R\&D$, not substantially different from the model developed by Funke and Strulik, but where there are however three essential refinements that we consider necessary to be made.

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1 Introduction

The balanced growth path of the endogenous growth model with physical capital, human capital and $R\&D$ has been explored first by Funke and Strulik (2000), and then by Arnold (2000). Funke and Strulik suggest that the typical advanced economy follows three development phases, characterized in a temporal order by physical capital accumulation, human capital formation, and innovation, and that the transitional dynamics of the model reproduce such a sequencing.

Few years later, Gomez (2005) analyzed the equilibrium dynamics of this model, generalize and correct the analysis of the papers of Funke and Strulik, and Arnold. Recently, Sequeira (2008) incorporates the erosion effect into an endogenous growth model in which growth is generated by the accumulation of physical, human capital, and $R\&D$, and claims this effect significantly improves the fit between the model and the data.

Iacopetta (2010) extends the earlier analysis of Funke and Strulik and argues that other sequences of the phases of development are possible and shows that the model can generate a trajectory in which innovation precedes human capital formation. This trajectory accords with the observation that the rise in formal education followed with a considerable lag the process of industrialization. It is important to be pointed out here that in a recent paper, Iacopetta (2011) introduces diminishing returns to time in $R\&D$, in order to broke the linearity of the Hamiltonian.

In our opinion, some results of these authors are affected by the fact that the Hamiltonian considered in these papers is linear (affine) in one of the control variables. Consequently, they are in the so called bang solution and from there it follows that there are only two possibilities. In the first case the control variable can take one of the two bounded limits of its interval of definition and in the second case it can take any value in that interval. Our paper tries to avoid this inconvenient by considering an alternative model. The outline of the paper is as follows. The first section is this introduction. In the second section we present a brief description of the model developed by Funke and Strulik, in the third section we develop an alternative model, in the fourth section we lay down some properties of the balanced growth path and the conditions for its existence, and in the final section we give some numerical simulations, compare our result with those of Funke and Strulik, and present some conclusions.

2 The model of Funke and Strulik

In this section, we summarize the model developed by FS and Arnold and derive the differential equations that describe the dynamics of the economy. The economy has a constant population of measure one. Each individual is endowed with one unit of time. A single homogenous final good Y is produced according to a CobbDouglas production function.

$$Y = A_1 K^{\beta} D^{\eta} (H_Y)^{1-\beta-\eta}, \tag{1}$$

where A is a positive constant, β and η are positive elasticity parameters with $\beta + \eta \leq 1$, K is physical capital, H_Y denotes the skills level of human capital employed in the final good sector and D represents an aggregate index of intermediate goods, the amount used for each one being $x(i), i = 1, 2, \cdots, n$. The market for final goods is perfectly competitive and the price for final goods is normalized to one, which implies a rental charge of $\beta Y/K$ for a unit capital. No-arbitrage requires that this rental charge equals the interest rate $r = \beta Y/K$ Furthermore, equating price and marginal production costs yields

$$p_D = \frac{Y}{D} \text{ and } w = (1-\beta-\gamma)\frac{Y}{H_Y},$$

where p_D represents the price index for intermediates. Each firm in the $R\&D$ sector owns an infinite patent for selling its variety $x(i)$. Producers act under monopolistic competition and maximize operating profits.

$$\pi(i) = [p(i) - 1]\, x(i),$$

where $p(i)$ denotes the price of an intermediate and 1 is the unit cost of Y. Facing the price elasticity of demand $\varepsilon = \frac{1}{1-\alpha}$ each firm charges a price $p = p(i) = \frac{1}{\alpha}$. Intermediate goods are embodied in the final product and hence depreciate completely with one cycle of production. One unit of each intermediate can be obtained from one unit of final output. Output is used for consumption and investment, and under symmetry $x(i) = x$. Hence we can write

$$D = x n^{\frac{1}{\alpha}} \tag{2}$$

and since

$$p_D \cdot D = pxn,$$

the total quantity of intermediates employed are given by

$$nx = \alpha\eta Y. \tag{3}$$

We assume that output of new intermediates is determined solely by the aggregate knowledge devoted to the $R\&D$ sector according to:

$$\dot{n} = \delta H_n, \tag{4}$$

with efficiency parameter $\delta > 0$. Additionally, individuals may spend part of their human capital, H_H on development of skills. This non-market activity is described by a production function of the Lucas (1988) type:

$$\dot{H} = \xi H_H \tag{5}$$

with efficiency parameter $\xi > 0$. The population size is normalized to one so that all aggregate magnitudes can be interpreted as per capita quantities. Human capital is supplied inelastically and leisure does not enter the utility function. Therefore, full employment requires:

$$H = H_Y + H_n + H_H. \tag{6}$$

Households earn wages w per unit of employed human capital $(H - H_H)$ and returns r per unit of aggregate wealth A, which leads to the budget constraint

$$\dot{A} = rA + w\,(H - H_H) - C. \tag{7}$$

If ν denote the value of an innovation, then free-entry into $R\&D$ requires

$$w = \delta\nu \text{ and } H_n > 0$$

in an equilibrium with innovation, or

$$w > \delta\nu \text{ and } H_n = 0$$

in an equilibrium without innovation. For simplicity, as in the paper of Funke and Strulik we neglect depreciation, which leads to the economy's resource constraint

$$Y = \dot{K} + C + nx. \tag{8}$$

In order to simplify the computational procedure we denote by $H_H = uH$, $H_n = vH$ and $H_Y = (1 - u - v)H$, with $u,\ v \in (0, 1)$ and $u + v < 1$. Subject to this constraint and to the knowledge formation technology they maximize intertemporal utility

$$\int_0^\infty \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} dt, \tag{9}$$

where $\rho > 0$ denotes the time preference rate and $0 < \theta^{-1} < 1$ defines the intertemporal elasticity of substitution. Using the state variables H and A and the control variables C and u, the Hamiltonian is defined as follows.

$$J(A, H, u, C) = \frac{C^{1-\theta}-1}{1-\theta} + \lambda \left[rA + w(1-u)H - C\right] + \mu\xi uH. \tag{10}$$

The first-order conditions are given by

$$\frac{\partial J}{\partial c} = 0 \;\; \Rightarrow \;\; c^{-\theta} = \lambda, \tag{11}$$

and

$$\frac{\partial J}{\partial u} = 0 \;\; \Rightarrow \;\; H(\xi\mu - w\lambda) = 0. \tag{12}$$

The derivatives of J with respect to H and A will produce

$$\frac{\dot{\lambda}}{\lambda} = \rho - r, \;\; \Rightarrow \;\; \frac{\dot{c}}{c} = \frac{r-\rho}{\theta} \tag{13}$$

and

$$\frac{\dot{\mu}}{\mu} = \rho - \xi. \tag{14}$$

Here is the big problem with the model of Funke and Strulik, because the hamiltonian is linear in the control variable u. Rearranging the terms of relation (10) we can write

$$J(A, H, u, C) = \frac{C^{1-\theta}-1}{1-\theta} + \lambda \left[rA + wH - C\right] + \left[\xi\mu - w\lambda\right] uH \tag{15}$$

The problem of maximizing intertemporal utility (9) subject the restrictions (5) and (7), is equivalent to the maximization of the Hamiltonian function J. One of the first order conditions implies

$$H(\xi\mu - w\lambda) = 0.$$

Since H is assumed to be positive but $\xi\mu - w\lambda \neq 0$ for all $t > 0$ this condition does not hold and we have only two possibilities for $u \in (0, 1)$, depending on the sign of $\xi\mu - w\lambda$. If $\xi\mu - w\lambda > 0$, then $u = 1$ and if $\xi\mu - w\lambda < 0$, then $u = 0$. (see Seierstad and Sydsaeter, pag. 165). If $\xi\mu - w\lambda = 0$ for all $t > 0$, then the Hamiltonian function given above does not depend on u and therefore it attaints its maximum for any value of $u \in (0, 1)$, or in other words, u is not a control variable. Surprisingly, the two authors assume that $\xi\mu = w\lambda$, from where it immediately follows

$$\frac{\dot{w}}{w} = \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda} = r - \xi,$$

indicating that the growth rate of wages must be sufficiently high compared to the interest rate to ensure investment in human capital. They also prove later that $\xi > \rho$. In fact they do not determine the optimal value of u because this is not possible. In what follows, the authors compute some equations that will be useful to characterize the three stages of development. The first is the resource constraint. Substituting (3) into the resource constraint (8) they arrive to the following equation

$$\dot{K} = (1 - \alpha\eta)Y - C. \tag{16}$$

Substituting (2) and (3) into the production function (1) and denoting by $H_Y = u_1 H$ they get

$$Y^{1-\eta} = A_2 K^{\beta} n^{\frac{(1-\alpha)\eta}{\alpha}} u_1^{1-\beta-\eta} H^{1-\beta-\eta} \tag{17}$$

where $A_2 = A_1 (\alpha\eta)^{\eta}$. Although these equations are not part of the optimal problem, together with equation (13) - derived from the optimal problem, can give a characterization of the case of developing economy, that means an economy capable of long run growth through accumulation of physical capital, expanding quantity of intermediate goods, and improvements in the quality of labor, but without developing new products. We remark here that the equation (32) at page 498, describing the trajectory of u_1 does not coincide with the same equation in the Lucas model, but has the same steady state value. This it happens because $\eta \neq 0$, even if they suppose the absence of development of new products, but at steady state, u_1 is independent of η. When Funke and Strulik analyze the case of the innovative economy, their equations (36) and (45) give the steady state value of u_1, where $H_Y = u_1 H$, but their model cannot produce the steady state value for the variable u, where $H_H = uH$. That is why we say that the results thereafter obtained by the authors, under this assumption, $\xi\mu - w\lambda = 0$ for all $t > 0$, are at least questionable. We do not claim that their results are incorrect.

3 An Alternative Model

In this section we develop an alternative model in order to obtain optimal solutions for all variables. Our model is not substantially different from the model developed by Funke and Strulik. There are however three essential refinements that we consider necessary to be made. The first consists in introducing the resource constraint in the optimal problem. In order to avoid the linearity of the Hamiltonian, we modify the definition of the variables H_Y and H_H, and finally we will consider only the case of the social planner's problem. Our model is characterized by the following optimization problem:

$$\int_0^\infty \frac{C^{1-\theta}-1}{1-\theta} e^{-\rho t} dt, \tag{18}$$

subject to

$$\begin{cases} \dot{K} = Y - C - nx, \\ \dot{H} = \xi H_H, \\ \dot{n} = \delta H_n, \\ H_0 = H(0),\ K_0 = K(0),\ n_0 = n(0), \end{cases} \tag{19}$$

where

$$Y = AK^\beta n^{\frac{\eta}{\alpha}} x^\eta H_Y^\gamma,$$

H is the human capital level of a representative worker. Consumption goods are produced competitively using human capital H_Y. In order to simplify the computational procedure we denote by $H_H = uH$ and $H_n = vH$, with $u, v \in (0,1)$. Consequently we have $H_Y = (1-u-v)H$. K, H and n are state variables and C, u, v and x are control variables, and $A > 0$, $\alpha > 0$, $\beta > 0$, $\eta > 0$, $\gamma > 0$, $\theta > 0$, $\rho > 0$, $\xi > 0$, with $\gamma = 1 - \beta - \eta$.
To solve the problem (18) subject to (19), we define the Hamiltonian function:

$$J = \frac{C^{1-\theta}-1}{1-\theta} + \lambda \left[AK^\beta n^{\frac{\eta}{\alpha}} x^\eta H^\gamma (1-u-v)^\gamma - C - nx\right] + \mu\xi H u + \nu\delta H v. \tag{20}$$

In an optimal program the control variables are chosen so as to maximize J. The necessary conditions for a maximum are:

$$\frac{\partial J}{\partial C} = 0 \ \Rightarrow\ C^{-\theta} = \lambda; \tag{21}$$

$$\frac{\partial J}{\partial u} = 0 \;\Rightarrow\; \gamma A K^{\beta} n^{\frac{\eta}{\alpha}} x^{\eta} (1-u-v)^{\gamma-1} H^{\gamma-1} \lambda = \xi \mu; \tag{22}$$

$$\frac{\partial J}{\partial v} = 0 \;\Rightarrow\; \gamma A K^{\beta} n^{\frac{\eta}{\alpha}} x^{\eta} (1-u-v)^{\gamma-1} H^{\gamma-1} \lambda = \delta \nu; \tag{23}$$

$$\frac{\partial J}{\partial x} = 0 \;\Rightarrow\; \left[\eta A K^{\beta} n^{\frac{\eta}{\alpha}} x^{\eta-1} (1-u-v)^{\gamma} H^{\gamma} - n\right] \lambda = 0; \tag{24}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial J}{\partial K} = \rho\lambda - \beta A K^{\beta-1} n^{\frac{\eta}{\alpha}} x^{\eta} (1-u-v)^{\gamma} H^{\gamma} \lambda; \tag{25}$$

$$\dot{\mu} = \rho\mu - \frac{\partial J}{\partial H} = \rho\mu - \gamma A K^{\beta} n^{\frac{\eta}{\alpha}} x^{\eta} (1-u-v)^{\gamma} H^{\gamma-1} \lambda - \xi u \mu - \delta v \nu; \tag{26}$$

$$\dot{\nu} = \rho\nu - \frac{\partial J}{\partial n} = \rho\nu - \left[\frac{\eta}{\alpha} A K^{\beta} n^{\frac{\eta}{\alpha}-1} x^{\eta} (1-u-v)^{\gamma} H^{\gamma} - x\right] \lambda. \tag{27}$$

The boundary conditions include initial conditions and the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)K(t) = 0,\; \lim_{t\to\infty} e^{-\rho t}\mu(t)H(t) = 0 \text{ and } \lim_{t\to\infty} e^{-\rho t}\nu(t)n(t) = 0.$$

From (22) and (23) it immediately follows that $\xi\mu = \delta\nu$. Substituting (22) into (26) we find

$$\frac{\dot{\mu}}{\mu} = \rho - \xi.$$

From (24) we deduce that

$$\eta A K^{\beta} n^{\frac{\eta}{\alpha}} x^{\eta} (1-u-v)^{\gamma} H^{\gamma} = nx, \tag{28}$$

that is $\eta Y = nx$ and substituting this result and (23) into (27) we get

$$\frac{\dot{\nu}}{\nu} = \rho - \frac{\delta\eta(1-\alpha)}{\alpha\gamma} \frac{H(1-u-v)}{n}.$$

Given the above relation concerning μ and ν we deduce that

$$\frac{H}{n} = \frac{\alpha\gamma\xi}{\delta\eta(1-\alpha)} \frac{1}{1-u-v}. \tag{29}$$

Substituting this result into the third equation of (19) we find that

$$\frac{\dot{n}}{n} = \frac{\alpha\gamma\xi}{\eta(1-\alpha)} \frac{v}{1-u-v}. \tag{30}$$

Taking the logarithm and differentiating (29) with respect to time we get

$$\frac{\dot{u}+\dot{v}}{1-u-v} = \xi u - \frac{\alpha\gamma\xi}{\eta(1-\alpha)}\frac{v}{1-u-v}. \tag{31}$$

Substituting (28) into the first equation of (19) it follows

$$\frac{\dot{K}}{K} = (1-\eta)\frac{Y}{K} - \frac{C}{K}. \tag{32}$$

Log differentiating (28) with respect to time we find

$$\beta\frac{\dot{K}}{K} + \left(\frac{\eta}{\alpha} - 1\right)\frac{\dot{n}}{n} - (1-\eta)\frac{\dot{x}}{x} - \gamma\frac{\dot{u}+\dot{v}}{1-u-v} + \gamma\frac{\dot{H}}{H} = 0,$$

and substituting the above results we get

$$\frac{\dot{x}}{x} = \beta\frac{Y}{K} - \frac{\beta}{1-\eta}\frac{C}{K} + \frac{\gamma\xi(\alpha\gamma+\eta-\alpha)}{\eta(1-\alpha)(1-\eta)}\frac{v}{1-u-v}. \tag{33}$$

Taking logarithm and differentiating (22) with respect to time we obtain

$$\beta\frac{\dot{K}}{K} + \frac{\eta}{\alpha}\frac{\dot{n}}{n} + \eta\frac{\dot{x}}{x} + (1-\gamma)\frac{\dot{u}+\dot{v}}{1-u-v} - (1-\gamma)\frac{\dot{H}}{H} + \frac{\dot{\lambda}}{\lambda} = \frac{\dot{\mu}}{\mu},$$

and substituting the above results we find the following equation

$$\frac{\dot{u}+\dot{v}}{1-u-v} = \frac{\xi[(1-\gamma)u-1]}{1-\gamma} + \frac{\beta}{(1-\gamma)(1-\eta)}\frac{C}{K}$$
$$-\frac{\gamma\xi(1-\alpha+\alpha\gamma)}{(1-\alpha)(1-\gamma)(1-\eta)}\frac{v}{1-u-v}. \tag{34}$$

Combining now (31) and (34) we get

$$\frac{C}{K} = \frac{\gamma\xi(\alpha\gamma-\alpha+\eta)}{\beta\eta(1-\alpha)}\frac{v}{1-u-v} + \frac{\xi(1-\eta)}{\beta} \tag{35}$$

and substituting this result into (33) we obtain

$$\frac{\dot{x}}{x} = \beta\frac{Y}{K} - \xi. \tag{36}$$

We can now close the system and write down the final form

$$\begin{cases} \frac{\dot{K}}{K} = (1-\eta)\frac{Y}{K} - \frac{\gamma\xi(\alpha\gamma-\alpha+\eta)}{\beta\eta(1-\alpha)}\frac{v}{1-u-v} - \frac{\xi(1-\eta)}{\beta}, \\ \frac{\dot{H}}{H} = \xi u, \\ \frac{\dot{n}}{n} = \frac{\alpha\gamma\xi}{\eta(1-\alpha)}\frac{v}{1-u-v}, \\ \frac{\dot{u}+\dot{v}}{1-u-v} = \xi u - \frac{\alpha\gamma\xi}{\eta(1-\alpha)}\frac{v}{1-u-v}, \\ \frac{\dot{C}}{C} = -\frac{\rho}{\theta} + \frac{\beta}{\theta}\frac{Y}{K}, \\ \frac{\dot{x}}{x} = \beta\frac{Y}{K} - \xi, \\ \frac{\dot{\lambda}}{\lambda} = \rho - \beta\frac{Y}{K}, \\ \frac{\dot{\mu}}{\mu} = \rho - \xi, \end{cases} \tag{37}$$

4 The balanced growth path

This section lays down the properties of the balanced growth path and the conditions for its existence. The system described above reaches the balanced growth path (BGP) if there exists t_* (possibly infinite), such that the set of functions of time $\{K(t), H(t), C(t), n(t), x(t), u(t), v(t)\}$ solve the optimal control problem and such that, for all $t \geq t_*$, $\{K(t), H(t), C(t), n(t), x(t)\}$ growth at constant rates r_K, r_H, r_C, r_n, r_x and $r_u = r_v = 0$. We denote by r_z the growth rate of variable z, z_* i its value at $t = t_*$ and z^* is its value for $t > t_*$. The following proposition gives our main result that characterizes the balanced growth path.

Proposition 1 *If $\alpha < \frac{\eta}{1-\gamma}$, $\xi > \rho$ and for all $t \geq t_*$, $r_u = r_v = 0$, then the above system reaches the BGP and the following statements are valid*

i. $r_{K*} = r_{C*}$ *with*

$$r_{K*} = \frac{\xi(\xi-\rho)\left[\eta(1-\alpha)+\alpha\gamma\right]}{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)}, \tag{38}$$

ii. $r_{H*} = r_{n*}$ *with*

$$r_{H*} = \frac{\alpha\gamma\xi(\xi-\rho)}{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)}, \tag{39}$$

iii.

$$r_{x*} = \frac{\xi\eta(1-\alpha)(\xi-\rho)}{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)}, \tag{40}$$

iv. $u_* \in [0,1], \; v_* \in [0,1]$

$$u_* = \frac{\alpha\gamma(\xi-\rho)}{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)}, \quad v_* = \frac{u_*(1-u_*)}{u_*+\varphi}, \tag{41}$$

$$1-u_*-v_* = \frac{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)-\alpha\gamma(\xi-\rho)}{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)+\eta(1-\alpha)(\xi-\rho)}, \tag{42}$$

v.

$$\frac{H_*}{n_*} = \frac{\alpha\gamma\xi}{\delta\eta(1-\alpha)}\frac{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)+\eta(1-\alpha)(\xi-\rho)}{\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)-\alpha\gamma(\xi-\rho)}, \tag{43}$$

$$\frac{C_*}{K_*} = \frac{\gamma\xi(\xi-\rho)\left[\alpha\gamma-\alpha+\eta\right]}{\beta\left[\alpha\gamma\theta+\eta(1-\alpha)(\theta-1)\right]}, \tag{44}$$

Proof of Proposition 1. Under the hypothesis $r_u = r_v = 0$, the fourth equation of the system (37) can be written

$$\xi\left[u - \frac{\varphi v}{1-u-v}\right] = 0, \; \varphi = \frac{\alpha\gamma}{\eta(1-\alpha)}$$

from where we deduce that

$$u^2-(1-v)u+\varphi v = 0 \quad \Leftrightarrow \quad v = \frac{u(1-u)}{u+\varphi}$$

whose solutions are given by

$$u_1 = \frac{1}{2}\left[1-v-\sqrt{(1-v)^2-4\varphi v}\right] \text{ and } u_2 = \frac{1}{2}\left[1-v+\sqrt{(1-v)^2-4\varphi v}\right].$$

What we need is

$$(1-v)^2-4\varphi v \geq 0$$

and since

$$0 \leq 1 + 2\varphi - 2\sqrt{\varphi^2 + \varphi} \leq 1,$$

the only acceptable condition is

$$v \in \left[0, 1 + 2\varphi - 2\sqrt{\varphi^2 + \varphi}\right].$$

Let v_* be such a solution and $u_* = u_{1*}$. It is just a simply exercise to prove that

$$1 - u_{1*} - v_* = u_{2*} \quad \Rightarrow \quad \varphi v_* = u_* u_{2*}.$$

Substituting this result into the third equation of the system (37) we get

$$r_n = \xi\varphi\frac{v_*}{u_{2*}} = \xi u_* = r_H$$

and consequently at BGP, H and n growth at the same constant rate r_H. The same property is obviously true for the variables Y and K. Let r_K be their common growth rate. Log differentiating $Y = AK^\beta n^{\frac{\eta}{\alpha}} x^\eta H^\gamma (1-u-v)^\gamma$ and knowing that $r_Y = r_n + r_x$ we have

$$(1-\beta)r_K = \left(\gamma + \frac{\eta}{\alpha}\right) r_H + \eta\left(r_K - r_H\right).$$

Finally we obtain

$$r_K = \frac{\eta(1-\alpha) + \alpha\gamma}{\alpha\gamma} r_H \quad \Rightarrow \quad r_x = \frac{\eta(1-\alpha)}{\alpha\gamma} r_H \quad \Rightarrow \quad \frac{Y}{K} = \frac{r_x + \xi}{\beta}.$$

From equation (35) we deduce that C/K is constant and therefore $r_C = r_K$. The fifth equation of the system (37) can be written

$$r_C = -\frac{\rho}{\theta} + \frac{\beta}{\theta}\frac{Y}{K} = -\frac{\rho}{\theta} + \frac{r_x + \xi}{\theta}$$

and hence we have

$$\begin{aligned}
&\frac{\eta(1-\alpha) + \alpha\gamma}{\alpha\gamma}\xi u_* = -\frac{\rho}{\theta} + \frac{\eta(1-\alpha)}{\alpha\theta\gamma}\xi u_* + \frac{\xi}{\theta} \\
\Rightarrow\ &\left[\frac{\eta(1-\alpha) + \alpha\gamma}{\alpha\gamma} - \frac{\eta(1-\alpha)}{\alpha\theta\gamma}\right]\xi u_* = \frac{\xi - \rho}{\theta} \\
\Rightarrow\ &u_* = \frac{\alpha\gamma(\xi - \rho)}{\alpha\gamma\theta + \eta(1-\alpha)(\theta - 1)} \quad \Rightarrow \quad v_* = \frac{u_*(1 - u_*)}{u_* + \varphi}.
\end{aligned}$$

All other results follows immediately by direct computation and thus the proof is completed.

5 Conclusions and numerical simulations

Our results obtained above, seem to be correct since if we put $\eta = 0$ we obtain exactly the results of the classical Lucas - Uzawa model (see Chilarescu (2011), pag. 110). A remark is absolutely necessary here. As we can see, our results do not coincide to those obtained by Funke and Strulik, first because the starting points are different. Also our results do not coincides with those obtained by Manuel Gomez from the same reasons. As we can see, in the case of decentralized solution, examined by Funke and Strulik and then by Manuel Gomez, we have the equality $nx = \alpha\eta Y$ and in the case of social planer solution we have $nx = \eta Y$.

In what follows we propose three benchmark values for our economy.

1. The first is those proposed by Funke and Strulik: $\beta = 0.36$, $\eta = 0.36$, $\alpha = 0.54$, $\xi = 0.05$, $\rho = 0.023$, $\theta = 2$, $\delta = 0.10$ and $A = 1$. The steady-state values are:

$$u_* = 0.0087,\ v_* = 0.0094,\ 1 - u_* - v_* = 0.9819,$$

$$r_{K*} = 0.0009,\ r_{H*} = 0.0004,\ r_{x*} = 0.0005,$$

$$\left(\frac{Y}{K}\right)_* = 0.1402\ \left(\frac{H}{n}\right)_* = 0.4649,\ \left(\frac{C}{K}\right)_* = -0.00006.$$

2. The second one is those proposed by Manuel Gomez: $\beta = 0.36$, $\eta = 0.36$, $\alpha = 0.40$, $\xi = 0.05$, $\rho = 0.023$, $\theta = 2$, $\delta = 0.10$ and $A = 1$. The steady-state values are:

$$u_* = 0.0069,\ v_* = 0.0130,\ 1 - u_* - v_* = 0.9801,$$

$$r_{K*} = 0.0010,\ r_{H*} = 0.0003,\ r_{x*} = 0.0007,$$

$$\left(\frac{Y}{K}\right)_* = 0.1407\ \left(\frac{H}{n}\right)_* = 0.2645,\ \left(\frac{C}{K}\right)_* = 0.00002.$$

3. We propose the following values: $\beta = 0.25$, $\eta = 0.25$, $\alpha = 0.35$, $\xi = 0.25$, $\rho = 0.04$, $\theta = 1.5$, $\delta = 0.10$ and $A = 1$. The steady-state values are:

$$u_* = 0.1069,\ v_* = 0.0807,\ 1 - u_* - v_* = 0.8124,$$

$$r_{K*} = 0.0515,\ r_{H*} = 0.0267,\ r_{x*} = 0.0248,$$

$$\left(\frac{Y}{K}\right)_* = 1.0993\ \left(\frac{H}{n}\right)_* = 3.3139,\ \left(\frac{C}{K}\right)_* = 0.0229.$$

As we can see, in the case of data used by Funke and Strulik, since $\alpha = 0.54$ and the necessary condition for existence of the BGP is $\alpha < 0.5$, the value of the ratio C/K is negative. In his paper, Manuel Gomez puts right a slight incorrectness in Funke and Strulik and Arnold papers.

A final remark is necessary here. We do not studied the stability properties of the BGP because of some difficulties of the computation procedure, but this will be realized in a future version of this paper.

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