Assessing technical efficiency in a divided production process framework. A directional non-parametric measure

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Abstract

Efficiency scores are usually based on distance computations to the frontier in an m+s–dimensional space, where m inputs produce s outputs. In addition, the efficiency improvements take into account the total consumption of each input. However, in many cases, each input can be divided into its own well–known stages, and trade–off among them is possible. If the sample firms share this framework, it can be used for computing efficiency. Such analysis provides information on the total optimal consumption of each input, as data envelopment analysis does, as well as on the most efficient assignment in each stage. This paper studies technical efficiency from this perspective. A non–parametrical methodology is presented, and a Multi-Criteria Linear Programming model (MLP) with input orientation is proposed. The analysis is performed considering different levels and defining the extent of satisfaction achieved at all these levels for each firm concerning its own utility function and weights for modelling firms' preferences. The satisfaction levels are computed regarding the frontier with better performance. Thus, MLP offers more detailed information to advise decision makers than other models proposed in the literature do. This detailed information could be useful and important to the decision–making team of each firm, and it can help them improve their performance. Numerical examples are also put forward to better illustrate the proposed method.

Keywords: efficiency and production processes, productivity and competitiveness, multicriteria and linear programming models, data envelopment analysis.

JEL: C02, C14, C44, C61.

1. Introduction

The most used methodologies for the efficiency estimation through the frontier function are the following: the mathematical programming by the data envelopment analysis or DEA (Cooper et al., 2007), and the so–called econometric frontier (Kumbahakar and Knox Lovell, 2000). The average efficiency level of the sample and the efficiency index of each firm can be estimated by using both methods.

Some no – parametrical methods have been proposed for estimating efficiency.They differ from conventional DEA standards, as the one proposed by Pastor and Aparicio (2010, 2011). The approach proposed in this paper also differs from conventional DEA standards. As far as we known, in the previous scientific literature, the technical efficiency of a sample of firms has been globally analyzed in a productive process, wherein m inputs X are consumed for producing s outputs Y, according to the technology given by the production frontier. The parametrical and non–parametrical methods provide efficiency scores which measure distances to the frontier, radially or directionally, in an $m + s$ dimensional space. In addition, both, the observed data and the efficiency improvement proposals are always related to the total consumption of each input: labor, capital, etc.

Nevertheless, there are companies with production processes where the assignment of some input X_i can be detached in its own well – known and specific stages and trade – off among such stages is feasible. An example of such input can be some non–specialized jobs. If all the firms of the sample have this detachment structure in common for some input X_i, then this fact can be very interesting to be considered as a basis for the efficiency analysis. Such analysis provides information on the total optimal consumption of each input, as DEA does, and also of the most efficient assignment in each stage.

This paper studies technical efficiency with this focusing. A new non–parametrical methodology is presented. Such methodology is applied to a productive process wherein the organizations of the sample present a common framework of detachment of each input in its own stages. That is, given an input, the stages in which such input is detached are the same for all the productive organizations, but can differ from the stages of detachment of the other inputs. A Multicriteria Linear Programming model (MLP) with input orientation is proposed.

The Multicriteria Decision making theory has, among others, a basic support in Linear Programming and the associated algorithms. The Multi-criteria Decision theory has its own and explicit theoretical framework, which includes well – specified conceptual definitions, and a systematic casual logic for assisting in decision making processes. Its goal is to reach a constrained optimization problem (model)in which the solution would lead to the best choice for the decision maker. The model assumes the existence of the decision – maker as an abstraction, enabling operational thinking (see, for example, Barba–Romero and Pomerol, 1997), and seeks to recreate (formalising and modelling) real–life situations. Multi-Criteria Linear Programming (MLP) plays an important role in the Multi-Criteria Decision theory.

Since the landmark work by Charnes and Cooper (1961), the common denominator for supporting the decision making has been to build an unified analytical structure of three basic components: attributes [*f(x)*], objectives [*Eff. f(x)*], and decision variables [*X*]; where *f(x)* stands for mathematical expression of attributes, and *Eff. f(x)* for finding efficient solutions (Romero, 1983). The general Multicriteria Linear-Programming (MLP) mathematical model can be written as follows:

$$\begin{aligned} &min\, f(X) = \left[f_1, \ldots f_i, \ldots f_k \right] \\ &s.t.\; X \in C, \end{aligned}$$

where $C \subseteq \mathbb{R}_+^n$ (decision space) is the non – empty feasible region, given as a vector non empty set of constrains that outlines the contour lines of possible solutions: $C = \left\{ X \in \mathbb{R}_+^n : AX \geq b \right\}$, being $b \in \mathbb{R}_+^m$ a data vector (the right side vector RHS as it is known in Linear Programming terminology), and $A \in \mathbb{R}_+^{m \times n}$ is the technical coefficients matrix. In the MLP context, it is necessary to define how the (vector) objective function should be assessed for different alternatives $X \in C$.

The dominance concepts and the ideas of dominated or non-dominated points are key aspects in MLP models in the so-called objective space (see, for example, Koomans, 1951). The non–dominated points are reference points for those that are dominated. Therefore, the control of resources in companies can be carried out by using technical efficiency scores as a standard procedure. In this line, this paper conducts a more detailed analysis (regards to each decision making unit (DMU), input and stage levels), than the existing standards in many DEA models. This detailed analysis is one of the main goals of the present paper.

All the different available techniques that enable the evaluation of the observed or hypothetical situations intend to improve the behavior and performance of the DMUs, by comparing them to a benchmark. In the case of MLP and DEA, both techniques provide an integrated framework for performance analysis in situations with homogeneous and independent DMUs. And both can be considered as complementary methods belonging to Multi-Criteria Decision Making (see, for example: Doyle and Green, 1993; Ray, 2004).

DEA models do not permit to incorporate decision makers' preferences, and no additional information is needed to reach the reference point. In fact, the intensities, or weights in which each DMU contributes to the efficient solution, are integrated into the model itself. Moreover, its input/output values are considered as targets for inefficient DMUs. On the other hand, several authors support that these targets are too restrictive when it comes to reducing the difference between them and the DMU evaluated, i.e. the difference between the projection obtained by the DEA and the input-output data. They try to include all inefficiencies the DEA model can identify (Pastor *et al*., 1999; Cooper *et al*., 2007; Pastor and Aparicio, 2011.

Korhonen *et al*. (2003) establish the correspondence between several radial DEA models and multi-objective linear-programming (MOLP) ones. They demonstrate that the use of the DEA radial projection to reach a reference point target is too restrictive. In fact, the presence of input and output slacks undermines the validity of a radial measure of technical efficiency. On the other hand, Joro *et al*. (1998) compare the structures of MLP and DEA. And they provided an interface between DEA and MLP through a Reference Point Model. This model projects each point on a reference frontier by weighting the vectors of inputs, reaching the frontier in a more flexible way than DEA does, i.e. sliding through the facets of the efficient frontier. Athanassopoulos (1995) puts together Goal Programming and DEA (GoDEA) to facilitate the development of a decision support system related to financial planning in local authorities in Greece. Suzuki *et al*. (2011), as an application to touristic regions in Italy, use the touristic production as a multiple-objective programming problem and the Euclidean distance minimization approach by the DEA. A survey of recent developments in MLP optimisation and applications can be found in Chinchuluum and Pardalos (2007). See also Saho and Ehrgott (2008) for an application in radiotherapy treatment planning.

A detailed analysis (from DMU to input and stage levels) is conducted in the present paper. This analysis requires the knowledge of the observed global output and input data, as well as the input data in each input stage. Firms' size is also considered in the MLP model. The analysis is applied for each DMU. It is formulated as a situation in which each DMU represents itself, having its own personalised efficient frontier, which is modelled according to a personal utility function. Such utility function is defined by the DMU itself according to its preferences. However, the structure of the production process is the same in all DMUs. The model is formulated in order to minimize the input-and input-stage consumptions. Efficiency scores, detailed to DMU level, input and stage levels, are also defined.

The paper is structured as follows. In the next section, the considered problem is justified and the main ideas for designing a production model are illustrated. It also shows the opportunity to assess a different technical efficiency score for each stage and each input. In Section 3, the MLP mathematical model is presented. Then, the MLP model is solved by the Utility Function Approach. The efficiency scores proposed are defined and computed. Finally, Section 3 defines the stage and input differential ratios, aiming at helping the decision-making team of each DMU. Section 4 applies the suggested MLP model to some numerical examples. Finally, some conclusions are drawn in Section 5, and the references are included in the final one.

2. Framework

This paper deals with the *technical efficiency analysis* of an economic sector. We are interested in evaluating the efficiency of a firm and its subsequent comparison to others in the same sector. The *m* inputs are used by the production process in every generic DMU (Decision Making Unit) in order to produce *s* outputs. In the economic sector under study, the set of tasks related to the *production process* can be detached in different *subsets*. In addition, *inputs* are to be detached in *stages, portions* or *dosages (doses),* and each part is assigned to the corresponding place in the production process.

In order to improve the efficiency of the production process, it is useful for the decision makers to be aware of the best performance options of its own DMU and how to reach them. It is also important to detect lacks, critical and weak points of the production process so as to find out where the improvements are required.

Unlike other authors, we suggest a model that allows us to *assess* a different technical efficiency score *for each stage and each input*.

For the above-mentioned aims of evaluation and comparison, the trend in the existing literature is to describe the activity of the firm by considering its inputs and outputs. Figure 1 also shows this idea. Therefore, given a list of *m inputs* ($x \in \mathbb{R}_{+}^{m}$), and *s outputs* ($y \in \mathbb{R}_{+}^{s}$), of a company or DMU, it is a common practice in economic analysis to describe the activities *(x, y)* that characterize the different firms through the *production set* of attainable points. This production set can be defined as:

$$\Psi = \left\{ (x, y) \in \mathbb{R}_{+}^{m+s} \; / \quad x \text{ can produce } y \right\} \qquad (1)$$

And it is also interesting to consider the so-called *input requirement set* for each output vector $y \in \mathbb{R}_{+}^{s}$ as the set of inputs than can produce such output, i.e.:

$$X(y) = \left\{ x \in \mathbb{R}_{+}^{m} / (x, y) \in \Psi \right\} \qquad (2)$$

To analyse technical efficiency of a company (DMU_o), these sets can be estimated on the basis of the observed values of a sample of *n* DMUs from the same economic sector. Obviously, these DMUs must be evaluated to estimate their efficiencies and compare them according to a specific DMU_o.

The most used non-parametric approach to solve this problem is the *Data Envelopment Analysis (DEA)*. For example, a typical DEA model such as the so-called *input oriented CCR model* can be applied for each DMU_0 to be evaluated. From this approach, the production possibility set (1) can be defined as follows:

$$P_{CCR} = \left\{ (x, y) \in \mathbb{R}_{+}^{m+s} \; / \; x \geq X\lambda, \; y \leq Y\lambda, \; \lambda \geq 0 \right\} \qquad (3)$$

Where $\lambda \in \mathbb{R}_{+}^{n}$ is a semipositive vector in $\mathbb{R}^{n}$, and the sample data of the different DMUs are arranged with the matrix of inputs $X \in \mathbb{R}_{+}^{m \times n}$ and the matrix of outputs $Y \in \mathbb{R}_{+}^{s \times n}$, wherein each column *j* of these matrixes, of input or output data, corresponds to the DMU_j of the sample data, with $j = 1, \ldots, n$.

The model for estimating such production possibility set was initially proposed by Charnes *et al*. (1978), and can be also seen in many publications and books as, for instance, Cooper *et al*. (2007). Such input-oriented CCR model, formulated in the *envelopment form*, can be written as:

Input-oriented CCR model:

$$\min_{\theta, \lambda} \theta \qquad (4)$$

s.t.:

$$\theta x_o - X\lambda \geq 0 \qquad (5)$$

$$Y\lambda \geq y_o \qquad (6)$$

$$\lambda \geq 0 \qquad (7)$$

Where $\lambda = (\lambda_1, ..., \lambda_n)^t \in \mathbb{R}^n$ is a vector, $X = (x_1, ..., x_n) \in \mathbb{R}^{m \times n}$ is an *m x n* matrix of inputs (with $x_j \in \mathbb{R}^m$ the data vector of the input values at DMU$_j$), and $Y = (y_1, ..., y_n) \in \mathbb{R}^{s \times n}$ is an *s x n* matrix of outputs (with $y_j \in \mathbb{R}^s$ the data vector of the output values at DMU$_j$).

The solution of the model (4) – (7) will provide us with an input-oriented *technical efficiency score* θ for DMU$_0$, which is a radial measure and identical for all *m* inputs. These scores lead to the comparisons among DMUs, and also offer decision makers useful information to improve efficiency. Nevertheless, they do not supply an estimation of the input-stage efficiency.

The model suggested presents estimations of the *global efficiency, input efficiencies,* and *stage efficiencies* for each DMU, input and stage. Moreover, *stage and input differential ratios* are defined to suggest improvements to DMUs.

3. The Proposed Mathematical Model

3.1 The model

Let us consider *m* inputs indexed by $i = 1, ..., m$; *s* outputs indexed by $r = 1,..., s$; and *n* DMUs indexed by $j = 1, ..., n$. DMUs develop a process consisting of several subsets of activities. The structure of the production process is identical in all DMUs. The inputs are different according to their nature, and the number of stages in which the different inputs can be detached. This idea is included in Figure 2, where the different nature of the inputs is represented by different grids.

For a given input *i*, $i = 1, ..., m$, this input can be detached into K_i stages indexed by $k = 1, ..., K_i$. We deal with situations where all DMUs have the same inputs, the same outputs, and (for each input) the same stages. Thus, *m, s* and K_i , $i = 1, ..., m$, are constants, and they do not depend on the considered DMU.

Let us define y_r^j as the value of output *r* in DMU$_j$, $r = 1,..., s$, $j = 1, ..., n$. In our production process, inputs are divided into stages; hence, x_{ik}^j is the value of input *i* in DMU$_j$ corresponding to stage *k*, $i = 1, ..., m$, $j = 1, ..., n$, $k = 1, ..., K_i$. For any given input *i*, we can summarize all the stages of such input in DMU$_j$, and then take

$$x_i^j = \sum_{k=1}^{K_i} x_{ik}^j \qquad (8)$$

as the total amount of input i in DMU$_j$, wherein K_i is used for denoting the number of stages in such input i, and $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Let us define the *production possibility set* for the production process that we are studying. Let us denote such production possibility set as P_{MLP}. The production possibility set is usually defined in the existing literature as stated in (1), without reaching the different input-stage levels of the different inputs. In this paper, we will reach the stage level.

Let $x = (x_{ik}) \in \mathbb{R}_{+}^{m\times K}$ denote a matrix of $m \times K$ input-stage quantities (with $K = \max_{i=1,\ldots,m}\{K_i\}$, and $x_{ik} = 0$ if $k > K_i$), and let $y \in \mathbb{R}_{+}^{s}$ denote a vector of s output quantities. The set of feasible combinations of input matrix and output vectors is given by the production possibility set

$$P_{MLP} = \{(x, y), x \in \mathbb{R}_{+}^{m\times K}, y \in \mathbb{R}_{+}^{s} \,/\, x \text{ produce } y\} \qquad (9)$$

The production possibility set P_{MLP} satisfies the following assumptions:

Assumption 1. The observed activities (x, y) belong to P_{MLP}.

Assumption 2. (x, y) does not belong to P_{MLP} if $x = 0$ and $y \geq 0$ with at least some output component strictly higher than 0. That is, it is not possible to produce any outputs without consuming inputs.

Assumption 3. For any activity (x, y) in P_{MLP}, any semi-positive activity $(\bar{x}, \bar{y})$ with $\bar{x} \geq x$ and $\bar{y} \leq y$ is included in P_{MLP}. That is, any activity with an input lower than x in any component *(i.e. entry of the input matrix)* and output no greater than y in any component *(i.e. entry of the output vector)* is feasible. That is, inputs and outputs are both strongly disposable. That is, the production possibility set P_{MLP} is a *FDH* (*free disposal hull*) production possibility set.

Assumption 4. For any activity (x, y) in P_{MLP}, *the ratio input / output* for each input i and output r is greater or equal to the minimum ratio input / output (for the corresponding input i and output r) among all the points in P_{MLP}. This takes place for each input i, $i = 1, \ldots, m$, and output r, $r = 1, \ldots, s$.

Assumption 5. For any activity (x, y) in P_{MLP}, *the ratio input / size* for each input i and stage k is greater or equal to the minimum ratio input / size (for the corresponding input i and stage k) among all the points in P_{MLP}. This takes place for each input i, $i = 1, \ldots, m$, and stage k, $k = 1, \ldots, K_i$.

Assumption 6. For any activity (x, y) in P_{MLP}, *the consumption of input* for each input i and stage k is non-negative, that is, greater or equal to zero. And it takes place for each input i, $i = 1, \ldots, m$, and stage k, $k = 1, \ldots, K_i$.

Then, our production possibility set is the set

$$P_{MLP} = \left\{ (x, y),\, x \in \mathbb{R}_+^{m \times K},\, y \in \mathbb{R}_+^{s} \,/\, (x, y)\ \textit{satisfies assumptions}\ 1\ \textit{to}\ 6 \right\} \quad (10)$$

Firstly, let us formally write the mathematical equations that define this production possibility set P_{MLP}. Later, in Subsection 3.1.3, we will propose an MLP model for estimating such production possibility set P_{MLP}.

The mathematical and formal expression of the activities of the production possibility set P_{MLP} are now presented. They can be formulated by a matrix x^P of inputs ($x^P \in \mathbb{R}_+^{m \times K}$, instead of a vector, which is the usual proceeding in the existing literature), and a vector y^P of outputs ($y^P \in \mathbb{R}_+^{s}$, as usual in the existing literature). The observed data of inputs at certain DMU_j with $j = 1, ..., n,$ can be arranged by a matrix $x^j = (x_{ik}^j) \in \mathbb{R}_+^{m \times K}$, with $K = \max_{i=1,\dots,m} \{K_i\}$, by displaying inputs in rows and input stages in columns, where each entry $x_{ik}^j \in \mathbb{R}_+$ of this matrix is some known and observed data representing the observed amount of input i at stage k of the production process in DMU_j, with $i = 1, ..., m,$ and $k = 1, ..., K, j = 1, ..., n,$ and assuming for notational convenience and without loss of generality that $x_{ik}^j = 0$ if $k > K_i$.

The *production possibility set* can be defined as

$$P_{MLP} = \left\{ \left(x^P, y^P\right),\, x^P \in \mathbb{R}_+^{m \times K},\, y^P \in \mathbb{R}_+^{s} \,/ \quad \begin{array}{c} x_i^P = \sum_{k=1}^{K_i} x_{ik}^P \geq y_r^P\, LB_{ir} \ , \forall\, i = 1,\dots,m \ , \forall\, r = 1,\dots,s \\ T_P\, l_{ik} \leq x_{ik}^P \ , \forall\, i = 1,\dots,m \ , \forall\, k = 1,\dots,K_i \\ 0 \leq x_{ik}^P \ , \forall\, i = 1,\dots m \ , \forall\, k = 1,\dots,K_i \end{array} \right\}$$

(11)

Where the lowest bounds LB_{ir} and l_{ik} are directly calculated from the data x_{ik}^j, x_i^j (defined by (8)), y_r^j, and T_j is a measure of the size of DMU_j, with $i = 1, ..., m, r = 1, ..., s, j = 1, ..., n, k = 1, ..., K_i$.

$$LB_{ir} = \min_{j=1,\dots,n} \left\{ \frac{x_i^j}{y_r^j} \,/\, y_r^j > 0 \right\}, \forall i = 1,\dots,m, \forall r = 1,\dots,s$$

(12)

$$l_{ik} = \min_{j=1,\dots,n} \left\{ \frac{x_{ik}^j}{T_j} \,/\, T_j > 0 \right\}, \quad \forall\, i = 1,\dots,m, \forall\, k = 1,\dots,K_i$$

(13)

And it is assumed that

$$T_P = \sum_{r=1}^{s} \tau_r y_r^P$$

(14)

reflects the size of the elements in P_{MLP}, characterized by a weighted mean of the outputs of each P_{MLP} element with known weights $\tau_r \geq 0$, $r = 1, \ldots, s$. These output weights are characteristic and specific constants of the production possibility set P_{MLP}.

The *first group of constraints* in (11) involves the total input/output ratios, which formulate Assumption 4, and are related to the consumed quantity of input i in the whole process at an arbitrary actual production unit. They establish that, in order to be realistic and to obtain feasible activities (x^P, y^P), i.e. pairs of input matrix x^P and output vector y^P, which can be really obtained, the ratio of the required input i divided by output r (ratio x_i^P / y_r^P), must be greater than the corresponding minimum ratio for this input i and output r, $i = 1, \ldots, m; r = 1, \ldots, s$. They reflect the fact that the total quantity of each input i to produce one unit of each output r cannot be lesser than the minimum amount consumed.

The *second group of constraints* in (11) concerns the required input amount at each input stage. They formulate Assumption 5. These constraints establish that the ratio of the required input i at each stage k divided by the firm size (ratio x_{ik}^P / T_P) must be greater than the corresponding minimum ratio for this input i and its stage k, $i = 1, \ldots, m; k = 1, \ldots, K_i$.

Finally, the *third group of constraints* in (11) means that there is no negative input levels for each input and input stage. They formulate Assumption 6 written above.

It is clear that this production possibility set P_{MLP}, defined in (11), also satisfies Assumptions 1, 2, and 3.

The boundary P_{MLP}^{∂} of the production possibility set P_{MLP} is given by:

$$P_{MLP}^{\partial} = \left\{ \left(x^P, y^P \right), x^P \in \mathbb{R}_+^{m \times K}, y^P \in \mathbb{R}_+^s \,/\, \text{at least one constraint of } P_{MLP} \text{ happens with equality} \right\}$$

(15)

The boundary P_{MLP}^{∂} of P_{MLP} constitutes the technology. The microeconomic theory of the firm suggests that, in perfectly competitive markets, firms operating in the interior of P_{MLP} will be driven by the market, but makes no prediction of how long this might take; moreover, a firm that is inefficient today might become efficient tomorrow. Note that the MLP model is also applicable even if markets are not perfectly competitive.

The MLP model estimates the production possibility set Ψ (conceptually defined and characterized in the existing literature usually by (1), as explained above) with P_{MLP} as (11). In addition, we will establish optimization criteria in order to go to the frontier P_{MLP}^{∂} given by (15) to achieve efficiency.

Given the input, size and output data; for each input *i*, we wish to optimize the total input amount of such input *i,* and the partial amounts of such input in all of its input stages, and this for each DMU, once. Hence, we need *n* optimizations, one for each DMU$_j$ to be evaluated. Each one of these optimizations is a *multi-criteria optimization problem*. As our model is input oriented, the objectives of these multi-criteria problems are to minimize inputs (in all input stages), but outputs are to be guaranteed. Note that the size of each DMU$_j$ is measured by parameter T_j. The MLP model works calculating such sizes according to (14), but MLP model could be also applied if these sizes are computed in a different way. Then, to minimize inputs in such a way that outputs are guaranteed is equivalent to minimizing input/size ratios guaranteeing outputs.

Let DMU$_j$ to be evaluated on any trial denote DMU$_o$, with $o = 1, \ldots, n$. Let us denote by K_i the number of stages of a given input *i*, $i = 1, \ldots, m$. For this DMU$_o$, the corresponding problem is the input-oriented multi-criteria linear-programming problem MLP_o showed below, where the *decision variables* are:

X_{ik}^{o} = desirable amount of input *i* in its input-stage *k* of the corresponding process at decision making unit DMU$_o$, for all inputs $i = 1, \ldots, m,$ and stages $k = 1, \ldots, K_i$. (16)

We can also summarize all stages of each input *i*, in DMU$_o$, by taking the decision variables:

$X_i^o = \sum_{k=1}^{K_i} X_{ik}^o$ desirable amount of input *i* in DMU$_o$, for all inputs $i = 1, \ldots, m$. (17)

And the MLP$_o$ problem is:

$$\min\left\{X_{1,1}^o, X_{1,2}^o, \ldots, X_{1,K_1}^o; X_{2,1}^o, X_{2,2}^o, \ldots, X_{2,K_2}^o; \ldots; X_{m,1}^o, X_{m,2}^o, \ldots, X_{m,K_m}^o\right\} \quad (18)$$

s.t.:

$$X_i^o \geq y_r^o \; LB_{ir} \quad \textit{for all inputs } i = 1, \ldots, m, \textit{ and outputs } r = 1, \ldots, s, \quad (19)$$

$$T_o \; l_{ik} \leq X_{ik}^o \quad \textit{for all inputs } i = 1, \ldots, m, \textit{ and stages } k = 1, \ldots, K_i, \quad (20)$$

$$0 \leq X_{ik}^o, \textit{for all inputs } i = 1, \ldots, m, \textit{ and stages } k = 1, \ldots, K_i, \quad (21)$$

Where the lowest bounds LB_{ir} and l_{ik} are defined in (12) – (13) for all inputs $i = 1, ..., m,$ outputs $r = 1, ..., s$, and input stages $k = 1, ..., K_i$, where T_o is a measure of the size of DMU$_o$, for $o=1,..., n$. For example, if there is only one output ($s = 1$, and $y_r^o = y^o$ for the only $r=1$), we can take $T_o = y^o$, the output value of DMU$_o$, $o=1,...,n$. If there are more outputs, T_o is estimated by:

$$T_o = \sum_{r=1}^{s} \tau_r y_r^o \qquad (22)$$

with known weights $\tau_r \geq 0$, $r = 1, ..., s$.

Note that, in our notation, capital letter X denote *decision variables* while non-capital x denote *observed input data*, i.e. known constants.

There are m inputs and, thus $\sum_{i=1}^{m} K_i$ criteria, the same number of criteria as stages considered. In fact, by (18) we are minimizing all stages of all the inputs in a multi-criteria approach. The constraints (19) – (21) estimate the production possibility set P_{MLP}. The model (18) – (21) tries to reach the boundary P_{MLP}^{∂} given by (15).

As any multi-criteria linear-programming problem, we can try to solve it by three different approaches (see for example: Zeleny, 1982; Romero, 1983; Yu, 1985; González-Martín, 1986; Steuer, 1986; Barba-Romero and Pomerol, 1997; Wallenius *et al.*, 2008): *(a) Utility function approach, (b) Hierarchical approach*, and *(c) Pareto optimality approach.*

The utility function approach consists in replacing the $\sum_{i=1}^{m} K_i$ criteria considered in (18) for a single criterion, usually a utility function depending on these criteria, before solving this resulting single criterion optimization problem with the constraints (19) – (21).

The hierarchical approach consists in, given a hierarchical priority among the $\sum_{i=1}^{m} K_i$ criteria considered in (18), solving firstly the single criterion problem given by the most important criterion and constraints (19) – (21); next, *among* the optimal solutions (or the set of solutions with the most important objective below a certain threshold), solving the single criterion problem given by the 2nd –the most important criterion and constraints (19) – (21), and so on.

The Pareto optimality approach consists in solving the problem (18) – (21) by means of Pareto optimality techniques for finding the so-called efficient frontier and Pareto optimal points. This efficiency and the Pareto optimal points must be understood from the point of view of the classic multi-criteria optimization. Approaches *(a)* and *(b)* just provide one point (or maybe more, but they would be fully equivalent) at the end in the *production possibility set*. Such point corresponds (and characterizes) the trial DMU$_o$ in the production possibility set. On the contrary, the Pareto optimal approach *(c)* can provide several non–dominated or Pareto optimal solutions for problem (18) – (21). These solutions are different and non–equivalent and they are shown by different points in the production possibility set. All these points together characterize DMU$_o$.

For simplicity, we only consider the approach *(a)* in this study, the Utility Function approach. The other approaches can be considered in future research.

3.2 Utility Function Approach

Let us consider the *utility function approach*, the main guidelines of which have been already specified to solve the *MLP* model. To establish such approach, we need, first of all, *the utility function*.

Therefore, we define a utility function. We choose a weighted linear combination of the criteria (inputs) considered in (18). Note that the weights are taken over every input at each stage, i.e. considering stage levels, and not only the total input.

Let us denote such utility function as

$$f(X^o) = \sum_{i=1}^{m} \sum_{k=1}^{K_i} w_{ik}^o X_{ik}^o \qquad (23)$$

where: X_{ik}^o = required amount of input i in the input stage k of the corresponding process in DMU$_o$, for all inputs $i = 1, ..., m,$ and stages $k = 1, ..., K_i$, as described in (16), that is, X_{ik}^o are the decision variables.

w_{ik}^o = weight or importance for input i and stage k regarding DMU$_o$, for all inputs $i = 1, ..., m,$ and stages $k = 1, ..., K_i$.

$X^o = (X_{ik}^o)$ $m \times K$ matrix of decision variables that characterize the solution space for DMU$_o$, with input index $i = 1, ..., m,$ and stage index $k = 1, ..., K$, where, for notational convenience, we are taking $K = \max_{i=1,...,m} \{K_i\}$ and assuming, without loss of generality, that $X_{ik}^o = x_{ik}^o = 0$ if $k > K_i$.

Then, our Utility Function Approach consists in solving the following

UF–MLP$_o$ model:

$$\min f(X^o) = \sum_{i=1}^{m} \sum_{k=1}^{K_i} w_{ik}^o X_{ik}^o \qquad (24)$$

s.t.:

(19) – (21).

Note that this utility function, expressed in (24), involves each DMU$_o$, and includes weights w_{ik}^o (for each input i and stage k; $i = 1, ..., m$; $k = 1, ..., K_i$). These weights are related to the decision variables X_{ik}^o, and they can differ in the different DMUs, i.e. each DMU has its own utility function. Note also that weights can be understood as preferences of the decision making units (DMUs), and they define the direction in which each DMU wishes to reach the frontier. The preferable direction for moving to the frontier will be the direction corresponding to the stage with a higher weight. In such stage, the solution will be the minimum value according to the technology.

3.3 Efficiency

Given the problem under study, the *production possibility set* is defined in Section 3.1.2 and estimated in Section 3.1.3. This section is devoted to define and compute different types of efficiency, with respect to DMU, input and stage. These efficiencies must be computed with regard to some reference (optimal) values, the target values. Our target values can be the optimal values of the objective functions of our multi-criteria linear-programming model (*MLP$_o$ model*). And, taking into account that a utility function approach is adopted to solve such multi-criteria model, i.e. we are working with a *UF–MLP$_o$ model*, we can directly take the optimal value of the objective function of such problem as our target value in order to compare, define and compute efficiency. Also note that the objective function defines a direction to the frontier P_{MLP}^{∂}, and this direction is defined with the weights w_{ik}^o, concerning DMU$_o$.

Thus, let us denote by X_{ik}^{o*}, for $i = 1, ..., m,$ and $k = 1, ..., K_i$ the optimal values of the decision variables resulting from solving the *UF–MLP$_o$* problem (problem (24) subject to (19) – (21)) at DMU$_o$. Note that alternative optimal solutions for this problem can be found. That is, it is possible for all the different values for X_{ik}^{o*}, for $i = 1, ..., m,$ and $k = 1, ..., K_i$, to be optimal. Even if there are alternative optimums for problem (24) subject to (19) – (21), the optimal value of the objective function will be identical, and is denoted by

$$Z^{o*} = \sum_{i=1}^{m} \sum_{k=1}^{K_i} w_{ik}^o X_{ik}^{o*} \qquad (25)$$

Definition 1. (Global efficiency for a given DMU): Given a DMU$_o$, with $o = 1, \ldots, n$, let us define the *global efficiency* of DMU$_o$, as:

$$\theta^o = \frac{Z^{o*}}{\sum_{i=1}^{m}\sum_{k=1}^{K_i} w_{ik}^o x_{ik}^o} \qquad (26)$$

This value is obviously bounded between 0 and 1, $0 \le \theta^o \le 1$. In addition, it is univocally defined with no ambiguity due to the uniqueness of Z^{o*}.

We say DMU$_o$ is *(globally) efficient* if its observed data (x_{ik}^o) satisfy $\theta^o = 1$. Otherwise, DMU$_o$ is *(globally) inefficient*.

Let us define now the *input efficiency of a given DMU$_o$ for each input i*, $i = 1, \ldots, m$. To do it we want to compute the best (minimum) value regarding the use of such input. Thus, for the given DMU$_o$, and for each input i, we previously consider the problem:

Problem input i – DMU$_o$. (Problem input i – UF – MLP$_o$)

$$\min Z_i^o = \sum_{k=1}^{K_i} w_{ik}^o X_{ik}^o \qquad (27)$$

s.t. (19) – (21) (restricted to the input i).

The constraints (19) – (21) restricted to the input i are a subset of the constraints (19) – (21). For completeness and a better understanding, we can formulate the following subset:

$$X_i^o \ge y_r^o \, LB_{ir} \quad \textit{for all outputs } r = 1, \ldots, s, \quad (28)$$

$$T_o \, l_{ik} \le X_{ik}^o \quad \textit{for all stages } k = 1, \ldots, K_i, \quad (29)$$

$$0 \le X_{ik}^o, \textit{for all stages } k = 1, \ldots, K_i, \quad (30)$$

Let Z_i^{o*} be the optimal value of the objective function of *problem input i – DMU$_o$*. This problem is solved independently from the problem of minimizing (24) s.t. (19) – (21), *problem UF-MLP$_o$*. The uniqueness of Z_i^{o*} is guaranteed even if there are alternative optimums to this new problem *input i – UF – MLP$_o$*. Let $\left\{X_{ik}^{o-(input\,i)*}\right\}$ (varying $k = 1, \ldots, Ki$) denote the coordinates of an optimal solution to the problem.

Definition 2. (Input efficiency for a given DMU in a given input i): Given DMU$_o$, with $o = 1, \ldots, n$, and an input i, $i = 1, \ldots, m$; let us define the *input efficiency of DMU$_o$ in such input i* as:

$$\theta_i^o = \frac{Z_i^{o*}}{\sum_{k=1}^{K_i} w_{ik}^o x_{ik}^o} \qquad (31)$$

This value is bounded between 0 and 1. Besides it is univocally defined with no ambiguity due to the uniqueness of Z_i^{o*}. And this uniqueness comes from the fact that Z_i^{o*} is the optimal value of the objective function of the *problem input i – UF – MLP$_o$*.

We say that DMU$_o$ is *input efficient in a given input i* if its observed data (x_{ik}^o) satisfy $\theta_i^o = 1$. Otherwise, DMU$_o$ is *input inefficient in such input i*.

Let us prove that, by solving problem (24) subject to (19) – (21), we have already automatically solved the *m* problems (27) subject to (28) – (30) indexed in the inputs *i*, *i = 1, ..., m*, and vice versa.

On the base of the above definitions we can establish the following obvious lemmas 1, 2 y 3.

Lemma 1: The feasible region of the problem *UF–MLP$_o$* is the disjoint union of the feasible regions of the problems *input i – UF – MLP$_o$,* with *i = 1, ..., m*.

Lemma 2: *m* problems *input i – UF – MLP$_o$,* with *i = 1, ..., m*, are independent.

Lemma 3: The objective function of the problem *UF–MLP$_o$* is exactly the sum of the objective functions of **the problems** *input i – UF – MLP$_o$,* with *i = 1, ..., m*.

Theorem 1

(a) To solve *problem UF–MLP$_o$* is equivalent to solving *m* independent *problems input i – UF – MLP$_o$,* with *i = 1, ..., m*.

(b) The optimal value of the objective function of *problem UF–MLP$_o$* is the sum of the optimal values of the objective functions of *m* independent *problems input i – UF – MLP$_o$,* with *i = 1, ..., m*.

(c) The set of coordinates (X_{ik}^{o*} , for *i = 1, ..., m,* and *k = 1, ..., K$_i$*) of any optimal alternatives in respect to *problem UF–MLP$_o$* consists in the (disjoint) union of the corresponding optimal set of coordinates regarding *m* independent *problems input i – UF – MLP$_o$,* with *i = 1, ..., m*.

Proof

See Appendix.

Corollary 1

Given a DMU$_o$, such DMU$_o$ is *(globally) efficient if, and only if, DMU$_o$ is input efficient in each input i for all inputs i = 1, ..., m.*

Proof

Obvious.

Let us define now the *stage efficiency of a given* DMU_o *in an input i (i = 1, …, m) for all stages k (k = 1, …,* K_i*)*. To do it we want to compute the best (minimum) value with regard to the use of the considered input i in such stage k. Thus, for the given DMU_o, for the input i, and for each stage k, we previously consider the problem:

Problem stage k – input i – DMU_o*. (Problem stage k – input i – UF –* MLP_o*)*

$$\min\ Z_{ik}^{o} = w_{ik}^{o} X_{ik}^{o} \qquad (32)$$

s.t. (19) – (21) (restricted to the input i).

Constraints (19) – (21) restricted to the input i are a subset of constraints (19) – (21), and this subset is defined by constraints (28) – (30). Taking into account that the weights w_{ik}^{o} are also known input data, such problem is equivalent to the problem:

$$\min\ X_{ik}^{o} \qquad (33)$$

s.t. (28) – (30).

More precisely, and to avoid any confusion, let us clearly differentiate *the considered stage k* from *all the set of stages* ρ*,* $\rho = 1, \dots, K_i$ of the given input i and, consequently, let us rewrite the constraints (28) – (30) in the way:

$$X_i^o \geq y_r^o\ LB_{ir} \quad \textit{for all outputs } r = 1, \dots, s, \ (34)$$

$$T_o\ l_{i\rho} \leq X_{i\rho}^{o} \quad \textit{for all stages } \rho = 1, \dots, K_i, \ (35)$$

$$0 \leq X_{i\rho}^{o}, \textit{for all stages } \rho = 1, \dots, K_i, \qquad (36)$$

Then, the problem is to minimize (33) subject to (34) – (36).

Note that the feasible region of this problem (about stage k of input i) is exactly the same that the feasible region of the problem corresponding to such input i, i.e. the *problem input i –* DMU_o. When we are solving the *problem stage k – input i –* DMU_o corresponding to the considered stage k, all the decision variables corresponding to all the stages of such input i are also present in the problem. In other words, we also consider all the other decision variables related to all the stages of input i, i.e. decision variables $X_{i\rho}^{o}$, *for all stages* $\rho = 1, \dots, K_i$. However, in the objective function there is only the decision variable associated to the stage k considered, variable X_{ik}^{o}. Note that this problem is solved independently from all the other problems previously considered.

Let $X_{ik}^{o-(input\ i,\ stage\ k)*}$ be the optimal value of the (single) decision variable present in the objective function of such *problem stage k – input i – DMU$_o$*. As the objective function has only one decision variable, then, even if there are several alternative optimal solutions, the optimal value $X_{ik}^{o-(input\ i,\ stage\ k)*}$ of such variable does not change. Furthermore, due to the way it is formulated, such optimal value is always at its lower bound, that is:

$$X_{ik}^{o-(input\ i,\ stage\ k)*} = T_o l_{ik} \quad (37)$$

Definition 3 (Stage efficiency of a given DMU$_o$ in an input i for a considered stage k)

Given a DMU$_o$, with $o = 1, \ldots, n$, and an input i, $i = 1, \ldots, m$; let us define for any stage k $(k = 1, \ldots, K_i)$ the *stage input efficiency of DMU$_o$ in such considered stage k of the input i* as:

$$\theta_{ik}^{o} = \frac{X_{ik}^{o-(input\ i,\ stage\ k)*}}{x_{ik}^{o}} = \frac{T_o l_{ik}}{x_{ik}^{o}} \qquad (38)$$

We can also name it *efficiency of the stage k in the input i in the DMU$_o$*.

This value is bounded between 0 and 1. Moreover, it is clear and univocally defined with no ambiguity because all optimal alternatives have the same value for the single decision variable present in the objective function.

This efficiency does not depend on the weights w_{ik}^{o}.

We say that DMU$_o$ is *stage efficient for a considered stage k corresponding to a given input i* if its observed data (x_{ik}^{o}) satisfy $\theta_{ik}^{o} = 1$. Otherwise, DMU$_o$ is *stage inefficient in such stage k corresponding to the given input i*.

Input efficiency and stage efficiency definitions are solutions of problems that have *exactly* the same constraints, that is, *the same feasible region*. They only differ in the objective function. The objective function of the, let us say, *"input"* problem is the weighted sum of the objective functions of the, let us say, *"stage"* problems. Consequently, the input problem is not separable in the stage problems and no equivalence among them exists. Nevertheless, a partial result can be established:

Theorem 2

If certain DMU$_o$ is *stage efficient for all the stages k of an input i*, then such DMU$_o$ is also *input efficient in such input i*.

Proof

For the direct, notice that all the problems mentioned in the Theorem statement have the same feasible region. The objective function of the input problem is a weighted sum of the objective functions of the so-called *"stage"* problems. Weights are no negative known data. Thus, if DMU$_o$ is simultaneously stage efficient at *all* the stages, then, by Definition 3, $x_{ik}^{o} = T_o \, l_{ik}$ *for all* $k = 1, \ldots, K_i$, and this solution satisfies (28) – (30). Besides, it is a solution of the common feasible region that (simultaneously) minimizes *all* the summands of the sum that defines the objective function of the "*input"* problem. The sum of minimums is feasible and, then, this sum of minimums is the minimum of the sum.

Corollary 2

The reciprocal is not necessarily true.

Proof

Trivial. Notice that, *it is not necessary* for the minimums of the different "stage" problems to be simultaneously held in the same feasible solution. In fact, such "ideal" point can be unfeasible (as usually happens in multi-objective problems). In this case, the minimum of the sum is not the sum of minimums because the minimums of the different summands are reached in different feasible solutions.

These ideas are illustrated by numerical examples in next section.

In this section, we have defined efficiency for each given DMU$_o$ at different levels: global efficiency, input efficiency, and stage efficiency. These measures provide an option for modelling and computing (for each given DMU$_o$) the *satisfaction level in resource consumption of such DMU$_o$ from all global, input and stage perspectives*.

If, for certain DMU$_o$, some of the different (global, input and/or stage) efficiencies reach the maximum value of 1, then it means that such DMU$_o$ works satisfactorily according to its own practical utilities and own purposes (modelled with its own weights and utility function) from the approaches (global, input, stage) where these scores are 1.

Otherwise, if a score fails and does not reach the maximum value of 1, this useful information can be used to point out where some efficiency improvements might be implemented. It could help to identify, at stage level, where a better management of the resource consumption is applicable.

Next section proposes some *differential ratios* that complement the efficiency scores previously defined and can help DMUs improve their performance.

3.4 Differential ratios

Given certain DMU$_o$, let $\left\{X_{ik}^{o*}\right\}$ (with *i = 1, …, m; k = 1, …, K*$_i$) be an optimal solution to problem (24) subject to (19) – (21). We can define the stage differential ratios and the input differential ratios as the ratios:

Stage differential ratio for stage k (k = 1, …, K$_i$*) of input i (i = 1, …, m):*

$$\Delta_{ik}^{o} = \frac{X_{ik}^{o*} - x_{ik}^{o}}{x_{ik}^{o}} \text{ for all inputs } i \text{ and stages } k \qquad (39)$$

Input differential ratio for input i (i = 1, …, m):

$$\Delta_{i}^{o} = \frac{X_{i}^{o*} - x_{i}^{o}}{x_{i}^{o}} = \frac{\sum_{k=1}^{K_i} X_{ik}^{o*} - \sum_{k=1}^{K_i} x_{ik}^{o}}{\sum_{k=1}^{K_i} x_{ik}^{o}} \text{ for all inputs } i \qquad (40)$$

These ratios represent *the proportion of reduction or increment suggested to DMU$_o$* to convert its current *(stage k, input i) data* x_{ik}^{o} to optimal value X_{ik}^{o*}; or to convert its current *input i – data* x_{i}^{o} to optimal value X_{i}^{o*}. These suggestions can help DMU$_o$ improve its current input data values.

These differential ratio definitions depend on the optimal solution considered. It is not a hard problem if the optimal solution $\left\{X_{ik}^{o*}\right\}$ (with *i = 1, …, m; k = 1, …, K*$_i$) is not identical unique. In this case, we can take any optimal solution among the set of optimal alternatives. To handle only one of these optimal solutions is enough for our purposes: to help DMU$_o$ managers enhance their objectives.

4. Conclusions

This paper proposes and presents a Multi-criteria Linear Programming model (MLP) with input orientation, which is aimed at computing efficiency scores of decision-making units (DMUs). The problem is motivated and the main requirements for the production process to be analysed are illustrated: the inputs are to be divided into stages in order to better modelling real-world situations.

The possibility to assess a different technical efficiency score for each stage and each input is also shown. The usual activity of the company is drawn by considering its inputs and outputs in the context of the input-oriented Data Envelopment Analysis problem (the so-called CCR model).

First, the production possibility set (P_{MLP}) is defined and then estimated by the MLP mathematical model which is proposed and defined. Three different approaches are pointed out to solve the multi-criteria problem resulting from the MLP model. Then, the MLP model is solved by the Utility Function Approach, which results from the weighted (preferences) linear combination of the criteria (inputs) for every input and stage. In this context we are dealing with a directional approach to assess technical efficiency.

The proposed efficiency scores are defined: global efficiency, input efficiency, and relations between both efficiencies are carried out for every DMU. Then, stage efficiency scores are established and also related to input efficiencies. The stage and input differential ratios are set in order to help decision-making teams of every DMU.

Among the main results and conclusions of the present research, we can highlight the following ones:

The MLP model is both an *input-oriented model* and an *input-stage oriented model*. This is because the MLP model is a multi-criteria linear programming model with a criterion for each pair *(input i, stage k)*, with $i = 1, \ldots, m$ and $k = 1, \ldots, K_i$; being m the number of inputs and K_i the number of stages into which such input i is divided.

The proposed MLP model carries out the analyses at a *global level,* just like other models previously presented in the existing literature, but it also differs from previous models, since it gets more in-depth analyses reaching *input and stage levels*.

This is a directional approach to estimate technical efficiency by means of a utility function that includes weighs, determining how to reach the frontier according to the decision makers' preferences.

Global and input efficiency scores provide the satisfaction levels in the consumption of inputs of such DMU. These measures are evaluated regarding the own utility function of the DMU, with its own set of weights to model the DMU's preferences. These satisfaction levels are gauged with respect to the frontier with the best performance. Whereas global efficiency considers the total input consumption, input efficiency is calculated focusing on just one input by a separated approach.

Stage efficiency score measures the proportion over 1 of the minimum feasible consumption for the stage considered and the current consumption in certain stage of a DMU, and do not depend on the set of weights.

In order to better advise decision makers, input and stage differential ratios show the possibilities to improve in input consumption, either at input or stage level.

As a global conclusion we can assert that the MLP model is a directional Multi-criteria Linear Programming model that is solved in a utility function framework and, therefore, with just one optimization criterion. It can be easily solved by the existing *Linear Programming* methods, software and tools. In addition, the constraints involved in the MLP model are, mainly, lower bounds of the decision variables; being the optimization problem a minimization one. These last features make the MLP model easier to solve, even if the number of constraints increases.

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Appendix

A. Proof of Theorem 1.

For proving the assertions *(a), (b)* and *(c)*, it is enough to prove that, known the coordinates of an optimal solution of the *problem UF–MLP*$_o$, automatically the coordinates of some optimal solutions of the *problems input i – UF – MLP*$_o$, with $i = 1, \ldots, m$, are known, and vice versa.

Let us prove it. For each input i, $i = 1, \ldots, m$, let $X_{ik}^{o-(input\,i)*}$ be a set of coordinates of certain optimal solution of the corresponding problem for such input i, *problem input i – UF – MLP*$_o$. Then, $Z_i^{o-(input\,i)*} = \sum_{k=1}^{K_i} w_{ik}^o X_{ik}^{o-(input\,i)*}$ is the optimal value of the objective function of such problem about such input i. Let us take $X_{ik}^o = X_{ik}^{o-(input\,i)*}$ for all i, $i = 1, \ldots, m$, and for all $k = 1, \ldots, K_i$. It is clear that this set of coordinates $\left\{X_{ik}^o\right\}$ (varying i and k) is a feasible solution of the *problem UF–MLP*$_o$. Even more, it is an optimal solution of such problem because if the sum of minimums is a feasible solution, such sum of minimums is the minimum of the sums. In other words, if $Z_i^{o-(input\ i)*} = \sum_{k=1}^{K_i} w_{ik}^o X_{ik}^{o-(input\ i)*} \leq \sum_{k=1}^{K_i} w_{ik}^o X_{ik}^{(o)}$ for all *i* and any other solution $\left\{X_{ik}^{(o)}\right\}$ (varying *i* and *k*), then $\sum_{i=1}^{m} Z_i^{o-(input\ i)*} = \sum_{i=1}^{m}\sum_{k=1}^{K_i} w_{ik}^o X_{ik}^{o-(input\ i)*} \leq \sum_{i=1}^{m}\sum_{k=1}^{K_i} w_{ik}^o X_{ik}^{(o)}$ for any other feasible solution $\left\{X_{ik}^{(o)}\right\}$ (varying *i* and *k*) of the *problem UF–MLP$_o$*. It means that the coordinates $X_{ik}^o = X_{ik}^{o-(input\,i)*}$ define an optimal solution of the *problem UF–MLP$_o$* and we can take $X_{ik}^{o*} = X_{ik}^{o-(input\,i)*}$ for all *i* and for all *k*.

Reciprocally, let $\left\{X_{ik}^{o*}\right\}$ (varying *i* and *k*) be an optimal solution of the *problem UF–MLP$_o$*. Let us take, for each input *i*, the point defined by $X_{ik}^{o-(input\ i)} = X_{ik}^{o*}$ (varying *k = 1, ..., K$_i$*). Like $\left\{X_{ik}^{o*}\right\}$ verifies (19) – (21), then $\left\{X_{ik}^{o-(input\ i)}\right\}$ is a feasible solution for the *problem input i – UF – MLP$_o$*, i.e., satisfies (28) – (30). Even more, it is an optimal solution of such problem by Lemmas 1, 2, and 3. Thus, we can take $X_{ik}^{o-(input\ i)*} = X_{ik}^{o*}$ as an optimal solution of the *problem input i – UF – MLP$_o$*, and it happens for all input *i*.

Then, trivially, *(a), (b)* and *(c)* hold.

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