Conjunctive Use of Groundwater and Rainwater: Supplement Irrigation and Implications for the resource

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Preliminary version

Abstract

Groundwater is typically extracted as a supplemental source to surface flows. However, today, groundwater table depletes at an alarming level whereas the demand is still increasing. Could Rainwater Harvesting (RWH) be a solution to water crisis? RWH could be a supplemental source of water and, above all, upgrade rainfed agriculture. However, storing rainwater prevents aquifer from recharging. Instead of saving groundwater, this technique contributes to its depletion.
We propose a dynamic model to outline this phenomenon. It is based on a trade-off between investment to store rainwater and groundwater extraction. By comparing this situation with an economy without investment, we conclude that the aquifer is not replenished.

Keywords: *Rainwater Harvesting, Conjunctive Use, Groundwater Management, Supplemental Irrigation, Water Crisis, Dynamic Model*
JEL: C12, C13, Q32, Q34

1 Introduction

The issue of groundwater management remains an important concern, especially in dry regions, and gets even more attention to the question of how to manage this resource. Surprisingly, it is used only as a complementarity source whereas it represents 30% of the Earth' water. Indeed, the main source of supplies comes from surface water representing only 0.3%. Because of increasing shortages in surface water, the pressure on groundwater rises such as in India where levels in many districts have fallen by more than 4 meters during 1981-2000. In fact, river water is fully used and Indian farmers have been trying to increase supplies by tapping underground reserves. Estimates show that farmers are pumping annually 100km^3 more that the monsoon rains replace.

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The fact that water tables plunge at an alarming rate incites strongly to change our approach for management. Traditionally, we used to deal with surface water and groundwater as separated entities as if they were unrelated system. However, this approach led to allocate twice the same amount of water and contributed to increase a great stress on the resource. At the opposite, the conjunctive management considers that surface water and groundwater are two manifestations of a single integrated resource. After the international conferences on water and environmental issues in Dublin and Rio de Janeiro held in 1992, scientists became aware of the importance of this concept. As a consequence, this approach reshapes the planning management where both resources are used in combination to improve availability and reliability. More precisely, this process relies on a shift between surface water and groundwater according to changes and shortages. While the most important supplies comes from surface water in wet season, the biggest part comes from groundwater extraction in dry season. Thus, groundwater is typically extracted as a supplemental source to surface flows to cope with peak demands or to meet deficits. This technique allows human to adapt and to extend water resources. However, since the population is still growing and dry seasons become more and more longer due to the climate change, people relies even more heavily on groundwater. Consequently, the risk of over-pumping and depletion is still increasing. Furthermore, the threat of a global water crisis may be accompanied by a food crisis. Indeed, food scarcity becomes the primary concern when water is scarce.

To this end, scholars focus attention on problem-solving technologies in order to preserve the remaining water sources and to improve the availability of water. This incentive is enforced by estimations indicating that about 25% of he increased water requirement needed to attain 2015 hunger reduction target of the Millennium Development Goal can be contributed from irrigation investment. Consequently, new methods appear to extend water resources. Amongst various solutions, we can quote recycling use water, brackish water desalination and ocean and rainwater harvesting. The two first methods are the most costly and only few countries access to these technologies . Rain Water Harvesting (abbreviated as RWH) is essentially collecting water. It has been attracted the attention of planners. In fact, this is not a new technique but a traditional one used by many old civilizations. India had been a pioneer in RWH methods. Namely, Rajasthan, a rain-deficient region in the northwest, is known for its innovative RWH systems. Many different definitions of rainwater harvesting have been given in the literature (Gould and Nissen-Petersen, 1999; Oweis and al. 1999, Frasier, 1994; Reij and al. 1988; Pacey and Cullis, 1986, Boers and Ben-Asher, 1982; Hollick, 1982; Dutt and al; 1981 Fraiser, 1975). In a broader sense, RWH is defined as the process of concentrating, collecting and storing water for different uses at a later time in a same area or in another one. For instance, the monsoon in India is gotten about 100 hours of rain in a year and it is this amount of water "that must be caught, stored and used over the other 8.660 hours that make up a year."[1] RWH is therefore used as a solution to overcome the problem of poor distribution in time by collecting rainwater when it rains and storing it for use to meet water needs in the preceding dry season. Thereby, this technique may be also

[1] this information is given by the website http://www.rainwaterharvesting.org/Solution/Solution.htm

an additional resource to cope with shortages as an aquifer does. Observational evidences witness that RWH has already been used as a supplemental source of water, especially in developing countries. Water tanks are very common in Gangsu province, in China. The Gangsu Research launched projects for water conservancy and by 2000, a total of 2,183,000 rainwater tanks had been built with a total of 73.1 million cubic meter supplying drinking water and supplementary irrigation. In middle of the nineties, Sri Lanka adopted the traditional wisdom of harvesting roof runoff because communities living the north central and southern dry areas had no access to safe drinking water on one hand and, on the other hand, the available groundwater was too brackish. In this case, RWH was an ideal solution to cope with shortages in surface water. In Saharan Algeria, a traditional system - *the foggaras* - transports rainwater to oasis through out the year. Concerning the developed world, even if the use of RWH is quite limited because of large-scale hydrologic project, there are some evidences of its application, namely in rural locations. In effect, the development of appropriate groundwater resources can be impractical for cost reasons (fewkes, 2006). Perrens (1982) estimates that in Australia approximately one million people rely on rainwater as their primary source of supply. RWH for potable use also occurs in rural areas of Canada and Bermuda (Fewkes, 2006).

Such examples illustrate how RWH can act as a substitute for other water sources. Several benefits have been listed to promote RWH. Namely, it is presented as an ideal solution to mitigate the over-exploitation of aquifers. But it permits also reducing energy consumption for lifting of groundwater. Estimates suggest that a one meter rise in water level saves about 0.40 KWH of electricity. Therefore, a trade-off between groundwater extraction and RWH may occur. A first contribution of this paper is to propose a dynamic model to study this trade-off. Furthermore, it is a matter of fact that there exists also an interrelationship between groundwater and rainwater. In effect, groundwater is primarily a depletable resource, although at a small proportion (5%) it can be withdrawn and renewed by seepage precipitations (Koundouri, 2004). As a consequence, a decreasing of seepage will reduce the replenishment. This study grapples with the question of how RWH will impact an aquifer.

This paper is organized s following. Section 2 proposes a simple groundwater extraction model. The dynamic model of groundwater management allows to outline some primary results. Section 3 yields an extension of the previous model by integrating a possibility to invest in capital. This capital provides a new source a water by storing the rainwater. The trade-off between both resource lead to observe an interesting result. Indeed, in this kind of model, the water table is lower than the water table in the simple groundwater management whereas the level of extraction is also lower. Section 4 concludes this analysis.

2 A simple model of groundwater extraction

We start from a dynamic, continuous time model of groundwater management for an aquifer with a constant and natural[2] recharge R. The upper-surface of groundwater

[2] A *natural recharge* results from snowmelt, precipitation or storm runoff. In this model, we rule out *artificial recharge*, i.e. the use of water coming from other sources to replenish the aquifer

is called the water table which can rise and fall in response to recharge and the rate at which water is extracted. Therefore the depth of the water table is significant in groundwater management. The depth of the water table has also an effect on whether a region will have a drought or not. The more the water table is deep, the less there is available groundwater. Assume that the depth measuring at period t is $d(t)$. Obviously, if $d(t) = 0$ then the water table reaches its maximum level and the aquifer is full. At the opposite, if the aquifer is totally empty then the depth reaches its maximum level denoted $\overline{d}$ and we have $\overline{d} - d(t) = 0$. Moreover, we assume the aquifer is characterized by a flat bottom and perpendicular sides. Therefore, the level of the water table is the same in each point of the aquifer. The dynamics of the water table across the time is given by the relation :

$$\dot{d} = R - w(t) \tag{1}$$

with $w(t)$ the amount of groundwater that is extracted at the period t.
A more realistic dynamics would have taken into account natural discharge, i.e. no other losses of water that may prevent drainage from reaching the water table and return flow to aquifer. Nevertheless, this simplification yields similar results.

Water is used in a production function as the lone variable input. The water social benefit derived from the use of groundwater is given by the production function, a strictly concave function :

$$f(w(t)), \;\; f'(w(t)) > 0, \;\; f''(w(t)) \leq 0 \tag{2}$$

The resource exploitation involves a cost depending on the amount that is pumped and on the level of the water table. Farzin (1996) proposes a similar formulation to take into account the current resource exploitation and a negative externality outlined by the cumulative amount exploited. In this study, this specification allows to take into account the fact the cost is pushed up by the intertemporal exploitation of the groundwater. Indeed, the use of the resource has an impact on the level of the water table and, according to this level, the effort to extract the groundwater will be more or less important. The cost is represented by a twice differentiable function :

$$C(w(t); d(t)), \;\; C_w > 0, \;\; C_d > 0, \; C_{ww} \geq 0, \; C_{dd} \geq 0 \; C_{wd} \geq 0, \; C_{dd}C_{ww} > (C_{wd})^2 \tag{3}$$

It is reasonable to assume that the marginal current exploitation cost will be higher both at higher exploitation amount and, for a constant extracted amount, at higher depth. Moreover, it is also reasonable to assume that the marginal cost due to the depth of the aquifer rises with the level of the depth.
We further assume that $C(0, 0) = 0$ and $C_w(0, 0) = C_d(0, 0) = 0$.

Therefore, the social net benefit of water consumption at period t is defined by :

$$f(w(t)) - C(w(t); d(t)) \tag{4}$$

We further assume that :

$$f_w(R) - C_w(R; 0) - \frac{1}{\rho} C_d(R; 0) > 0 \tag{5}$$

$$f_w(R) - C_w(R; \overline{d}) - \frac{1}{\rho} C_d(R; \overline{d}) < 0 \tag{6}$$

The assumption 5 means that as long as the aquifer is full, there is a marginal social net benefit to exploit an additional unit of the resource when we withdraw the recharge. The assumption 6 means that it is too costly to totally withdraw the groundwater when we have already extracted the recharge.

2.1 The Model

The social planner will choose the optimal extraction path maximizing the total present values of social welfare. Formally, the social planner's problem is given by :

$$\max_{w_t} \int_0^\infty \Big[f(w(t)) - C(w(t); d(t))\Big] e^{-\rho t} dt \tag{7}$$

$$\text{subject to} \begin{cases} \dot{d} = w(t) - R, \ \ d(0) = d_0, \ \ d(\infty) \text{ free} \\ g_1\left(w(t)\right) \geq 0 \\ g_2(d(t)) = d(t) \geq 0 \\ g_3(d(t)) = \overline{d} - d(t) \geq 0 \end{cases}$$

This is a generalized programm with one mixed constraint depending on the control variable and two pure state constraints. Taking into account all the constraints leads to the following Lagrangian :

$$\begin{aligned} L\left(w(t), d(t), p(t), q(t), \mu_1(t), \mu_2(t)\right) &= \widetilde{L}\left(w(t), d(t), p(t), q(t)\right) - \mu_1(t)\left(w(t) - R\right) \\ &+ \mu_2(t)\left(R - w(t)\right) \end{aligned} \tag{8}$$

where

$$\widetilde{L}\left(w(t), d(t), p(t), q(t)\right) = H(w(t), d(t), p(t)) + q(t)w(t) \tag{9}$$

and

$$H(w(t), d(t), p(t)) = f(w(t)) - C(w(t); d(t)) + p(t)\left(w(t) - R\right) \tag{10}$$

The variable $q(t)$ represents the lagrangian multiplier associated with the mixed constraint $g_1(w(t))$ and Both μ_1 and μ_2 are non-decreasing variables associated to each pure state constraint, i.e. $g_2(d(t))$ and $g_3(d(t))$. Finally, $p(t)$ is the co-state variable. An usual assumption consists of saying that the presence of pure state constraints involve a discontinuity in the co-state variable. Therefore, we can assume that $p(t)$ is piecewise continuous and piecewise continuously differentiable with jump discontinuities at a finite number of points τ_k such as $k = 1, \ldots, N$ and $t_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_N \leqslant t1$. The variable β_j is a number using to define a jump.

2.1.1 Necessary conditions

The solution of problem 7 has to satisfy the following almost necessary conditions stated in accordance with theorem 9 and note 6 in Seierstad and Sydsaeter (1987, respectively p.381 and p.375).

$$\begin{aligned}
\widetilde{L}_w &= f_w(w_t) - C_w(w(t); d(t)) + p(t) + q(t) = 0 && (11)\\
q(t)w(t) &= 0,\ \ q(t) \geq 0 && (12)\\
\mu_j & \quad \text{constant on any interval where } g_j(d(t)) > 0 \ \ j = 2;3 \\
\mu_j & \quad \text{continuous on any interval } t \in]t_0, t_1[\text{ at which } g_j(d(t)) = 0 \\
& \quad \text{and } \partial_d \centerdot g_j(w(t) - R) \text{ discontinuous } j = 2;3 && (13)\\
\dot{p}(t) &= \rho p(t) - \widetilde{L}_d = \rho p(t) + C_d(w(t); d(t)) && (14)\\
p(t) & \quad \text{can have a jump discontinuity such as} \\
p(\tau_k^-) - p(\tau_k^+) &= \beta_{1k} \frac{\partial g_1(d(t))}{\partial d} + \beta_{2k} \frac{\partial g_2(d(t))}{\partial d} && (15)\\
\text{with } \beta_{jk} &\geq 0 \, (= 0 \text{ if } g_j > 0) \ \ j = 2;3
\end{aligned}$$

In an *a priori* guess, this framework allows various regimes to occur whether or not the constraints are binding. To this end, a first step consists of studying the possibility of each regime in order to delete those that are impossible. Two of them are obviously impossible. Indeed, the constraint $g_2(d(t))$ cannot be simultaneously binding with the constraint $g_3(d(t))$. In other words, the depth of the water table cannot be simultaneously equal to zero and to its maximum level. Both constraints are incompatible whatever the level of extraction. In line with this first remark, we propose the following lemmas.

Lemma 1 $\forall t \in]t_0, t_1[$, *the mixed constraint is never binding. An amount of groundwater is always used,* $w(t) > 0$.

Proof 1 Assume that $w = 0$ *then the dynamics 1 becomes :* $\dot{d} = -R$. *If we resolve this differential equation according to the initial condition* $d(0) = d_0$, *we find that* $d(t) = d_0 - Rt$. *But,* $\lim_{t \to +\infty} d(t) = -\infty$ *This result contradicts the fact that* $\overline{d} \geq d(t) \geq 0$ ■.

From this lemma, we insure that the lagrangian multiplier $q(t)$ is always equal to zero, that satisfies condition 12. Moreover, by combining this result with the property of the partial derivative of the production function, we insures also that the production sector is always active.

The two following lemmas deal with the pure state constraints.

Lemma 2 $\forall t \in]t_0, t_1[$, *the aquifer is never full such as* $d(t) = 0$.

Proof 2 Since $d(t) = 0$ *then* $\dot{d} = 0$ *and equation 1 is written such as :* $R = w(t)$ *given lemma 1. We can rewritten equation 11 :* $p(t) = C_w(R; 0) - f_w(R)$. *The co-state variable is therefore constant* $\forall t \in]t_0, t_1[$ *and we have* $\dot{p} = 0$. *By computing this result with 11 and 14, we obtain that* $f_w(R) - C_w(R; 0) - \frac{1}{\rho} C_d(R; 0) = 0$. *This result contradicts the assumption 5* ■.

The symmetric regime can be deleted in the same way.

Lemma 3 $\forall t \in]t_0, t_1[$, *the aquifer is never empty such as* $d(t) = \overline{d}$.

Proof 3 Since $d(t) = \overline{d}$ *then* $\dot{d} = 0$ *and equation is written such as :* $R = w(t)$ *given lemma 1. We can rewritten equation 11 :* $p(t) = C_w(R; \overline{d}) - f_w(R)$. *The co-state variable is therefore constant* $\forall t \in [t_0, t_1]$ *and we have* $\dot{p} = 0$. *By computing this result with 11 and 14, we obtain that* $f_w(R) - C_w(R; \overline{d}) - \frac{1}{\rho} C_d(R; \overline{d}) = 0$. *This result contradicts the assumption 6* ■.

Lemma 2 and 3 show that the pure state constraints are inactive and, therefore, we know that the aquifer is never full or totally empty at the optimum. The combination of these three lemmas lead to a lone possible regime corresponding to the interior solution.

Proposition 1 $\forall t \in [0, \infty[$, *(i) By lemmas 2 and 3, condition 13 is clarified by saying that* μ_j *is constant over any interval and condition 16 gives* $p(t)$ *continuous. (ii) Moreover, given conditions 14-16 and lemmas 1-3, the optimum is the interior solution.*

By proposition 1, some necessary conditions can be rewrite in order to give some interpretations. By rewriting condition 11, we obtain an expression of the co-state variable.

$$p(t) = C_w(w(t); d(t)) - f_w(w_t) \tag{16}$$

The shadow price of the resource is equal to the difference between the marginal production cost and the marginal productivity of the resource. This specification is counter-intuitive comparing with the standard formulation in the literature. Indeed, by reasoning with the stock of groundwater, we used to get that the marginal benefit is equal to the sum of the marginal extraction cost and the opportunity cost of removing one unit of water from the ground. This scarcity rent reflects the effect on the profit when the stock is diminished. However, with the expression of our dynamics, if the depth is decreasing then the stock of water is increasing. Rearranging equation 14 yields :

$$\dot{p} - C_d(w(t); d(t)) = \rho p(t) \tag{17}$$

Equation 17 is a standard optimal condition that must hold for all t. The left-hand side is the marginal benefit of conservation. The first term is the forgone increase in value that would have been realized by conserving the marginal unit. The second term is the increase in future cost due to extracting the marginal unit now instead of later. The right-hand side is the forgone marginal benefit of extracting water in terms of monetary value realized after one period. This is the interest rate on the resource shadow price. We obtain therefore the result that, at the margin, the benefit extracting water must equal the cost of extraction.

2.1.2 Sufficient conditions

We turn now to a study of sufficient conditions. According to the Mangasarian's sufficient conditions in infinite horizon, an admissible pair $(d^*(t), w^*(t))$ must satisfy some conditions to be an optimum. According to the theorem 11 in Seierstad and Sydsaeter (1987, .385), we can state these conditions as follows :

$$w^*(t) \text{ maximizes } \widetilde{L}(d^*(t), w(t), p(t)) \tag{18}$$

$$\dot{p} - \rho p(t) = -\widetilde{L}_d \tag{19}$$

$$q_j(t) \geq 0 \; (0 \text{ if } g_j\,(d^*(t), w^*(t) > 0) \;\; j = 1, \ldots, 3 \tag{20}$$

$$g_j\,(d(t), w(t)) \;\text{ quasi-concave in }\; (d(t), w(t)) \;\; j = 1, \ldots, 3 \tag{21}$$

$$H(d(t), w(t), p(t)) \text{ is concave w.r.t}(d(t), w(t)) \tag{22}$$

$$p(\tau_k^-) - p(\tau_k^+) = \sum_{j=1}^{3} \beta_{jk}^+ \frac{\partial g_j\left(d^*(\tau_k), w^*(\tau_k^+)\right)}{\partial d} + \sum_{j=1}^{3} \beta_{jk}^- \frac{\partial g_j\left(d^*(\tau_k), w^*(\tau_k^-)\right)}{\partial d} \tag{23}$$

$$\text{where for all } k = 1, \ldots, N \;\; \beta_{jk}^+ \cdot \frac{\partial g\left(d^*(\tau_k), w^*(\tau_k^+)\right)}{\partial w} = 0 \tag{24}$$

$$\beta_{jk}^- \cdot \frac{\partial g\left(d^*(\tau_k), w^*(\tau_k^-)\right)}{\partial w} = 0 \tag{25}$$

$$\beta_{jk}^+ \geq 0 \; (0 \text{ if} g_j\left(d^*(\tau_k), w^*(\tau_k^+) > 0\right) \;\; j = 1, \ldots, 3 \tag{26}$$

$$\beta_{jk}^- \geq 0 \; (0 \text{ if} g_j\left(d^*(\tau_k), w^*(\tau_k^-) > 0\right) \;\; j = 1, \ldots, 3 \tag{27}$$

$$\lim_{t \to +\infty} \left(p(t) \exp^{-\rho t}(d(t) - d^*(t))\right) = 0 \text{ for all admissible } d(t) \tag{28}$$

First of all, notice that the sufficient conditions are quite different from the necessary conditions because the lagrangian variables $q_j(t)$ and the numbers $\beta_j(t)$ are defined for all constraints and not only for the mixed constraint and, respectively, the pure state constraints. Then, it is easily shown that the other conditions are satisfied. Indeed, the two first conditions represents the maximum principle. With lemmas 1-3, we know that all constraints are inactive on over any interval. Then, by 20, 26 and 27, we obtain $q(t) = 0$, $\beta_{jk}^+ = 0$ ad $\beta_{jk}^- = 0$. So, from 23, $p(t)$ is continuous. The constraints function g are linear in $(d(t), w(t))$ so 21 is trivially satisfied. Finally, we show that :

$$\begin{vmatrix} H_{ww} & H_{wd} \\ H_{dw} & H_{dd} \end{vmatrix} = \begin{vmatrix} f_{ww} - C_{ww} & -C_{wd} \\ -C_{dw} & -C_{dd} \end{vmatrix} = -f_{ww}C_{dd} + C_{wd}C_{dd} - C_{wd}C_{dw} > 0$$

Therefore, the hamiltonian is concave.

Proposition 2 *The interior solution satisfies the sufficient conditions and then, is the optimum.*

2.2 Steady state and stability

The last step is to outline the steady state. First, we reduce the dimensionality of the model from three variables $(w(t), d(t), p(t))$ to two variables $(w(t), d(t))$. Taking the time derivative of the shadow price in equation 11 and using this result, equation 11 itself and equation 14 yields :

$$\dot{w} = \frac{(w(t) - R)C_{wd} - \rho[C_w - f'(w(t))] - C_d}{f''(w(t)) - C_{ww}} \tag{29}$$

Thus, we obtain the time derivative of the resource extraction. Therefore, equations 1 and 29 define a system of two differential equations whose solution gives the optimal time paths for $w(t)$ and $d(t)$.

The steady state is investigated by setting the time derivatives 1 and 29 respectively equal to zero. Equation 1 does not depend on d(t) and yields the steady state for the extraction rate directly : $w = R$.
On the other hand, we have from equation 29 :

$$\left.\frac{dd}{dw}\right|_{\dot{d}=0} = -\frac{RC_{wd} + \rho[C_{ww} - f''(w(t))] + C_{dw}}{C_{dd}} < 0 \tag{30}$$

since $C_{dd} \geq 0, C_{ww} \geq 0, C_{dd} \geq 0, C_{wd} \geq 0$ and $f'' \leq 0$.
The slope of the isocline $\dot{w} = 0$ is therefore negative. Hence we have to verify that this isocline cuts the isocline $\dot{d} = 0$. In other words, we have to verify the existence and the uniqueness of the equilibrium.

Lemma 4 *There exists a unique steady state.*

Proof 4 It is straightforward that there exists a solution :

$$\lim_{d\to 0} \big(f'(R) - C_w(R, d(t)) - C_d(R, d(t))\big) > 0$$

with assumption 5.
Because $\frac{dd}{dw} < 0$*, the solution is unique.*

Now, we characterize the steady state. From equation 29, we have :

$$f'(R) = C_w(R, d^*) + \frac{1}{\rho} C_d(R, d^*) \tag{31}$$

This condition establishes that marginal benefit must be equal to the total marginal cost of production. The total cost is the sum of the marginal cost of extraction and the opportunity cost of removing one additional unit from the aquifer. In other words, this expression means that the marginal benefit when we extract the natural recharge must be equal to the marginal cost of extraction and the capitalized value of the increase in marginal cost resulting from a reduction in groundwater stock equal to the natural recharge. This condition represents a standard result equalizing the marginal benefit from using a natural resource with the marginal cost. The second term in the left-hand side represents the marginal user cost of the resource. As usual, we verify that this cost is increasing with a decreasing amount of available freshwater.

Now, we determine the types of this steady state. We linearize the system around the steady state and analyze the resulting Jacobian matrix. Letting J_{mn} denote an element in row m and column n of the Jacobian matrix, the system given by 1 and 29 implies the following elements :

$$\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{d}}{\partial d} & \frac{\partial \dot{d}}{\partial w} \\ \frac{\partial \dot{w}}{\partial d} & \frac{\partial \dot{w}}{\partial w} \end{pmatrix} \tag{32}$$

with $J_{11} = 0$, $J_{12} = 1$,

$$J_{21} = \frac{\rho(C_{wd}+C_{dd})(C_{ww}-f'')}{(f''(w(t))-C_{ww})^2} > 0 \text{ and } J_{22} = \frac{(-RC_{wd}-\rho(C_{ww}-f'')-C_{dw})(f''-C_{ww})}{(f''(w(t))-C_{ww})^2} > 0.$$

Hence, the sign of the determinant of the Jacobian matrix is clear. It is negative and therefore, since the determinant is equal to the product of the characteristics roots, we can deduce that the two roots have opposite signs which establishes that the critical point is locally a saddle point.

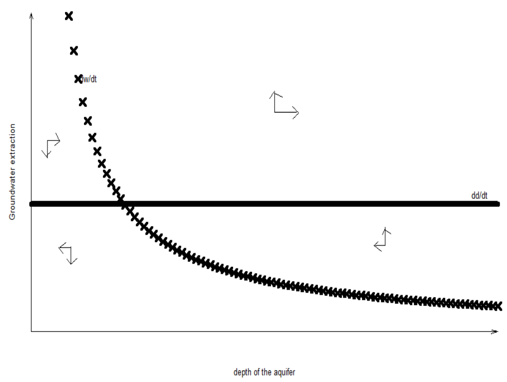

Figure 1: The dynamics of the simple groundwater model

Proposition 3 ***A global Saddle-point***
According to the global saddle point theorem of Seierstad and Sydsaeter (p.256), there exists an optimal trajectory to the system defined by equation 1 and 29 such that $d^*(0) = d_0$, $d^*(t) \to d_e$ *and* $w(t) \to w_e$ *as* $t \to \infty$.

Proof 5 To demonstrate this proposition, we have to verify two conditions :
(i) $J_{12}J_{21} > 0$
(ii) Rename $F = \dot{d} = w(t) - R = 0$ *and* $G = \dot{w} = 0$. *There exists positive D and B such that :*

$$\left|\frac{G}{F}\right| \leq D|w| \text{ for all } |w| \geq B$$

where $D = \frac{(w(t)-R)[\rho(C_w - f_w + C_d]}{C_{ww} - f_{ww}}$

This proposition allows us to conclude that the interior solution converge to the equilibrium from the initial condition $d(0)$.

3 Groundwater extraction and investment

We now extend the model to incorporate a new source of water in the production function. Farmers can use groundwater and rainwater through their level of investment. In this model, the production results from the combination of two substitutable inputs : the groundwater $w_g(t)$ and water coming from capital $w_s(t)$. As previously, we have a positive marginal production but evolves at diminishing rate :

$$f\left(w_g(t)+w_s(t)\right),\ \ f_{w_g}, f_{w_s}>0,\ \ f_{w_g w_g}, f_{w_s w_s}\leq 0 \tag{33}$$

Moreover, these inputs are perfectly substitutable therefore there are not impact on the level of the production.

$$f_{w_g w_s}<0 \tag{34}$$

This new water source impacts the dynamics 1 of the groundwater. Indeed, the recharge is now reduced by the amount of water that can be collected through the capital stock. Therefore, the motion law of the groundwater is written as follows :

$$\dot{d}=w_g(t)-(R-w_s(t)) \tag{35}$$

The capital stock varies also across time and can be increased by investing, where the investment rate is denoted by $I(t)$. As usual, capital stock increases with investments and decreases with depreciation. Assuming that the capital is depreciated according to a convex function :

$$\delta(w_s(t)), \delta(0)=0;\ \delta'\geq 0, \delta'(0)=0\ \delta''\geq 0, \tag{36}$$

The dynamics of the capital stock across time is given by the relation :

$$\dot{w}_s=I(t)-\delta(w_s(t)) \tag{37}$$

Besides the purchase costs, there is a cost of investment adjustment which assumed to be strictly convex :

$$\Theta(I(t)),\ \Theta(0)=0;\ \Theta'\geq 0,\ \Theta'(0)=0;\ \Theta''>0 \tag{38}$$

Further, we assume :

$$f'(R)-MCC(R)<0 \tag{39}$$

with MCC the marginal cost of capital.
This assumption means that the marginal cost of capital is higher than the marginal benefit when we extract the recharge.
At this moment, we leave undefined the marginal cost of the capital and we are going to specify it more precisely later.

The social planner will choose the optimal paths of groundwater and investment maximizing the total present values of the social welfare. Formally, the problem becomes :

$$\max_{w(t),I(t)}\int_0^\infty \Big[f(w_g(t)+w_s(t))-C(w_g(t);d(t))-\Theta(I(t))\Big]e^{-\rho t}dt \tag{40}$$

$$\text{subject to}\ \begin{cases}\dot{d}=w_g(t)+w_s(t)-R,\ \ d(0)=0\\ \dot{w}_s=I(t)-\delta(w_s(t))\\ g_1 w_g(t)\geq 0\\ g_2(d(t))=d(t)\geq 0\\ g_3(d(t))=\overline{d}-d(t)\geq 0\\ g_4(w_s(t))=w_s(t)\geq 0\end{cases}$$

where the extraction cost $C(w_g(t); d(t))$ has the same properties described in 3.

This is a generalized programm with one mixed and three pure state constraints. Taking into account all the constraints leads to the following modified Lagrangian :

$$L\left(w_g(t), w_s(t), d(t), p_1(t), p_2(t), q(t), \mu_1(t), \mu_2(t), \mu_3(t)\right) = \widetilde{L}\left(w(t), w_s(t), d(t), p_1(t), p_2(t), q(t)\right)$$
$$-\mu_1(t)(w_g(t) + w_s(t) - R) + \mu_2(t)(R - w_g(t) - w_s(t)) - mu_3(t)(I(t) - \delta(w_s(t))) \quad (41)$$

where

$$\widetilde{L}\left(w_g(t), w_s(t), d(t), p_1(t), p_2(t), q(t)\right) = H\left(w_g(t), w_s(t), d(t), p_1(t), p_2(t), q(t)\right) + q(t)w_g(t) \quad (42)$$

and

$$\begin{aligned} H\left(w_g(t), w_s(t), d(t), p_1(t), p_2(t)\right) &= f(w_g(t) + w_s(t)) - C(w_g(t), d(t)) - \Theta(I(t)) \\ &+ p_1(t)(w_g(t) - R) + p_2(t)(I(t) - \delta(w_s(t))) \end{aligned} \quad (43)$$

The co-state variables $p_1(t)$ and $p_2(t)$ are piecewise continuous and piecewise continuously differentiable with jump discontinuities at a finite number of points τ_k such as $k = 1, \dots, N$ and $t_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_N \leqslant t1$. The variable $q(t)$ represents the lagrangian multiplier and β_j is a number using to define a jump.

3.1 Necessary Conditions

We use the same methodology as previously and according to theorem 9 and note 6 in Seierstad and Sydsaeter (1987, respectively p.381 and p.375), we obtain the following almost necessary conditions :

$$\widetilde{L}_{w_g} = \partial_{w_g} f(w_g(t) + w_s(t)) - \partial_{w_g} C(w_g(t); d_t(t)) + p_1(t) + q(t) = 0 \quad (44)$$

$$q(t)w_g(t) = 0 \;\; q(t) \geq 0 \quad (45)$$

$$\widetilde{L}_I = -\partial_I \Theta(I(t)) + p_2(t) = 0 \quad (46)$$

μ_i constant on any interval where $g_j(d(t)) > 0$ $i = 1 \dots 3$ $j = 2; 3; 4$

μ_i continuous on any interval $t \in]t_0, t_1[$ at which $g_j(d(t)) = 0$

and $\partial_d g_j \cdot (w(t) - R)$ discontinuous $j = 2; 3$

μ_3 and $\partial_{w_s} g_4 \cdot (I(t) - \delta(w_s(t))$ (47)

$$\dot{p}_1(t) = \rho p_1(t) - \widetilde{L}_d = \rho p_1(t) + \partial_{d_t} C(w_g; d_t) \quad (48)$$

$$\dot{p}_2(t) = \rho p_2(t) - \widetilde{L}_{w_s} = \rho p_2(t) - \partial_{w_s} f(w_g + w_s) - p_1(t) + p_2(t)\partial_{w_s}\delta(w_s) \quad (49)$$

$p_1(t)$ and $p_2(t)$ can have a jump discontinuity such as

$$p_1(\tau_k^-) - p_1(\tau_k^+) = \beta_{1k}\frac{\partial g_1(d(t))}{\partial d} + \beta_{2k}\frac{\partial g_2(d(t))}{\partial d} \quad (50)$$

$$p_2(\tau_k^-) - p_2(\tau_k^+) = \beta_{3k}\frac{\partial g_3(w_s)}{\partial w_s} \quad (51)$$

with $\beta_{jk} \geq 0$ ($= 0$ if $g_j > 0$) (52)

As previously, this framework allows various regimes to occur whether or not constraints are binding. We delete directly all the regimes for which we have the depth of the aquifer simultaneously equal to zero and to its maximum level. In a same way, we focus on the constraint $g_4(w_s(t))$ and we can guess that it is never binding.

Lemma 5 *Over any interval of time (i) where we do not use the groundwater, i.e.* $w_g(t) = 0$*, and (ii) where we use groundwater, i.e.* $w_g(t) > 0$*, it is impossible to do not use rainwater, i.e.* $w_s(t) = 0$.

Proof 6 We show by contradiction the two part of the lemma.
$\forall\, t \in]t_0, t_1[$*, when* $w_s(t) = 0$*, then* $\dot{w}_s = 0$ *which implies that* $I(t) = \delta(0) = 0$*. Then, from 46, we obtain* $p_2(t) = 0$ *with the assumption that* $\partial_I C(0) = 0$ *and therefore,* $\dot{p}_2 = 0$*. From 49, we obtain that* $p_1(t) = -\partial_{w_s} f[w_g(t)]$.
For both parts of the lemma, it possible to identify 3 cases: either the aquifer is full, or empty of at a level between both boundaries.
(i) Turn to the first part,

- *Over any interval of time such that* $d(t) = 0$ *or* $d(t) = \overline{d}$*, we know that* $\dot{d} = 0$*. With* $(w_g; w_s) = (0; 0)$ *then* $R = 0$*. This contradicts the fact that* $R > 0$
- *Over any interval of time such that* $\overline{d} > d(t) > 0$*, we obtain that* $\dot{d} = -R$*. By integrating, we obtain* $\int_0^\infty \dot{d}dt = -\int_0^\infty Rdt \Leftrightarrow d(t) = -Rt$*. Since* $R > 0$*, this contradicts the fact that* $d(t) > 0$

(ii) Turn to the second part,

- *Over any interval of time such that* $d(t) = 0$ *or* $d(t) = \overline{d}$*, we know that* $\dot{d} = 0$*. Therefore,* $w_g(t) = R$ *and* $p_1(t) = -\partial_{w_s} f(R)$*. We deduce that* $\dot{p}_1 = 0$*. From 48 we obtain a different definition for* p_1.
- *Over any interval of time such that* $\overline{d} > d(t) > 0$*, we obtain from 44 that* $p_1(t) = \partial_{w_g} C[w_g(t); d(t)] - \partial_{w_g} f[w_g(t)]$ *that contradicts* $p_1(t) = -\partial_{w_s} f[w_g(t)]$

■.

Lemma 6 *Over any interval of time such as* $w_s(t) > 0$ *and* $d(t) = 0$ *or* $d(t) = \overline{d}$*, we have* $w_g(t) > 0$.

Proof 7 We show by contradiction that $w_g(t) > 0$ *even if* $w_s(t) > 0\ \forall d(t) = \{0; \overline{d}\}$.
Whenever $d(t) = 0$ *or* $d(t) = \overline{d}$ *then* $\dot{d} = 0$*. By assuming that* $w_g(t) = 0$*, we obtain that* $w_s(t) = R$ *and therefore* $\dot{w}_s = 0$ *which implies that* $I(t) = \delta(R)$*. By differentiating 46, we obtain* $\dot{p}_2 = 0$ *and computing this result with 49, we obtain* $\dot{p}_1 = 0$.
(i) If $d(t) = 0$*, from 48, we obtain that* $p_1 = -\frac{1}{\rho}\partial_{d(t)} C(0; 0) = 0$ *with the assumption* $\partial_{d(t)} C(0; 0) = 0$*. By computing* $p_1 = 0$ *and equation 44, we find that* $q = -\partial_{w_g} f(R)$*. Since* $\partial_{w_g} f(R) > 0$*, this result contradicts the fact that* $q \geq 0$*. ■. (ii) If* $d(t) = \overline{d}$*, from 48, we obtain that* $p_1(t) = -\frac{1}{\rho}\partial_{d(t)} C(0; \overline{d}) = 0$*. We compute this result into equation 49 and we obtain* $\frac{1}{\rho}\partial_{d(t)} C(0; \overline{d}) = \partial_{w_s} f(R) - p_2(t)$*. From 46, we know that* $p_2(t) = \partial_I \Theta\left(\delta(R)\right)$ *therefore* $\frac{1}{\rho}\partial_{d(t)} C(0; \overline{d}) = \partial_{w_s} f(R) - \partial_I \Theta\left[\rho + (\delta(R))\right]$*. The second*

term of the right-hand side represents the marginal cost of capital.
Since the right-hand side is negative under our assumption 39, this contradicts the fact that $\frac{1}{\rho}\partial_{d(t)}C(0;\overline{d}) > 0$. ■.

Lemma 7 *Over any interval of time such as* $w_s(t) > 0$ *and* $w_g(t) = 0$*, we have* $\overline{d} - d(t) > 0$.

Proof 8 Over any interval of time such that $d(t) = \overline{d}$*, we know that* $\dot{d} = 0$*. Therefore,* $w_s(t) = R$ *and* $\dot{w}_s = 0 \Leftrightarrow I(t) = \delta(R)$*. From 46, we have* $p_2(t) = \Theta'(\delta(R)) \Rightarrow \dot{p}_2 = 0$*. From 49, we find that :* $[\rho + \delta'(R)]\Theta'(\delta(R)) - f'(R) = -\frac{1}{\rho}C_d(0,\overline{d})$*. Under our assumption 39, we know that* $[\rho + \delta'(R)]\Theta'(\delta(R)) - f'(R) > 0$ *which contradicts the fact that* $\frac{1}{\rho}C_d(0,\overline{d}) < 0$. ■.

These lemmas bring additional information for the optimal solution. From lemma 5 and 6, we conclude that the constraint $g_4(w_s(t))$ is never binding and therefore, according to 52, the number associated to a possible jump is $\beta_3 = 0$. The co-state variable $p_2(t)$ is continuous. Because of the following remaining regimes, we deduce that there is two possible jumps for the co-state variable $p_1(t)$: either a jump associated to the constraint $g_2(d(t))$ or to the constraint $g_3(d(t))$.

Table 1: Possible regimes

Regime	q	$d(t)$	$\overline{d} - d(t)$
Full Aquifer	0	0	> 0
Empty Aquifer	0	> 0	0
Interior case	0	> 0	> 0

By studying these last regime, we find that only the interior solution can be a steady state. Therefore, the three others are possible trajectories.

3.1.1 Sufficient conditions

We turn now to a study of sufficient conditions. According to the Mangasarian's sufficient conditions in infinite horizon, an admissible pair $(d^*(t), w_g^*(t), w_s^*(t))$ must satisfy some conditions to be an optimum. According to the theorem 11 in Seierstad and Sydsaeter (1987, .385), we can state these conditions as follows :

$$w_g^*(t), I^*(t) \text{ maximizes } \widetilde{L}(d^*(t), w_g^*(t), w_s^*(t), p_1(t), p_2(t)) \tag{53}$$

$$\dot{p_1} - \rho p_1(t) = -\widetilde{L}_d \tag{54}$$

$$\dot{p_2} - \rho p_2(t) = -\widetilde{L}_{w_s} \tag{55}$$

$$q_j(t) \geq 0 \; (0 \text{ if } g_j \left(d^*(t), w_g^*(t), w_s^*(t) > 0\right) \;\; j = 1, \ldots, 4 \tag{56}$$

$$g_j \left(d(t), w_g(t), w_s(t)\right) \text{ quasi-concave in } \left(d(t), w_g(t), w_s(t)\right) \;\; j = 1, \ldots, 4 \tag{57}$$

$$H(d(t), w_g^*(t), w_s^*(t), p_1(t), p_2(t)) \text{ is concave w.r.t}(d(t), w_g(t), w_s(t)) \tag{58}$$

$$p_1(\tau_k^-) - p_1(\tau_k^+) = \sum_{j=1}^{4} \beta_{jk}^+ \frac{\partial g_j \left(d^*(\tau_k), w_g^*(\tau_k^+), w_s^*(\tau_k^+)\right)}{\partial d} + \sum_{j=1}^{4} \beta_{jk}^- \frac{\partial g_j \left(d^*(\tau_k), w_g^*(\tau_k^-), w_s^*(\tau_k^-)\right)}{\partial d} \tag{59}$$

$$p_2(\tau_k^-) - p_2(\tau_k^+) = \sum_{j=1}^{4} \beta_{jk}^+ \frac{\partial g_j \left(d^*(\tau_k), w_g^*(\tau_k^+, w_s^*(\tau_k^+)\right)}{\partial w_s} + \sum_{j=1}^{4} \beta_{jk}^- \frac{\partial g_j \left(d^*(\tau_k), w_g^*(\tau_k^-), w_s^*(\tau_k^-)\right)}{\partial w_s} \tag{60}$$

$$\text{where for all } k = 1, \ldots, N \;\; \beta_{jk}^+ \cdot \frac{\partial g \left(d^*(\tau_k), w_g^*(\tau_k^+, w_s^*(\tau_k^+)\right)}{\partial w_g} = 0 \tag{61}$$

$$\text{where for all } k = 1, \ldots, N \;\; \beta_{jk}^+ \cdot \frac{\partial g \left(d^*(\tau_k), w_g^*(\tau_k^+, w_s^*(\tau_k^+)\right)}{\partial I} = 0 \tag{62}$$

$$\beta_{jk}^+ \cdot \frac{\partial g \left(d^*(\tau_k), w_g^*(\tau_k^-, w_s^*(\tau_k^-)\right)}{\partial w_g} = 0 \tag{63}$$

$$\beta_{jk}^- \cdot \frac{\partial g \left(d^*(\tau_k), w_g^*(\tau_k^-, w_s^*(\tau_k^-)\right)}{\partial w_s} = 0 \tag{64}$$

$$\beta_{jk}^+ \geq 0 \; (= 0 \text{ if} g_j \left(d^*(\tau_k), w_g^*(\tau_k^+, w_s^*(\tau_k^+)) > 0\right) \;\; j = 1, \ldots, 4 \tag{65}$$

$$\beta_{jk}^- \geq 0 \; (= 0 \text{ if} g_j \left(d^*(\tau_k), w_g^*(\tau_k^-, w_s^*(\tau_k^-)) > 0\right) \;\; j = 1, \ldots, 4 \tag{66}$$

$$\lim_{t \to +\infty} \left(p_1(t) \exp^{-\rho t}(d(t) - d^*(t))\right) = 0 \text{ for all admissible } d(t) \tag{67}$$

$$\lim_{t \to +\infty} \left(p_2(t) \exp^{-\rho t}(w_s(t) - w_s^*(t))\right) = 0 \text{ for all admissible } w_s(t) \tag{68}$$

$$\tag{69}$$

First of all, notice that the sufficient conditions are quite different from the necessary conditions because the lagrangian variables $q_j(t)$ and the numbers $\beta_j(t)$ are defined for all constraints and not only for the mixed constraint and, respectively, the pure state constraints. Indeed, the two first conditions represents the maximum principle. With lemmas 5 and 6, we know that the mixed constraint $g_1(w_g(t))$ and the pure constraint $g_4(w_s(t))$ are inactive on over any interval. Then, by 56, 65 and 66, we obtain $q_1(t) = 0$ and $q_4(t) = 0$ on one side, and on the other side, we get $\beta_{1k}^+ = 0$, $\beta_{1k}^- = 0$, $\beta_{4k}^+ = 0$ and $\beta_{4k}^- = 0$. So, from 60, $p_2(t)$ is continuous. All the constraints function g_j are linear in $(d(t), w_g(t), w_s(t))$ so 57 is trivially satisfied. Finally, we show that :

$$\begin{vmatrix} H_{w_gw_g} & H_{w_gw_s} & H_{w_gd} \\ H_{w_gw_s} & H_{w_sw_s} & H_{w_sd} \\ H_{dw_g} & H_{dw_s} & H_{dd} \end{vmatrix} = \begin{vmatrix} f_{w_gw_g} - C_{w_gw_g} & f_{w_gw_s} & -C_{w_gd} \\ f_{w_sw_g} & f_{w_sw_s} - p_2(t)\delta_{II} & 0 \\ -C_{dw_g} & 0 & -C_{dd} \end{vmatrix}$$
$$= f_{w_gw_g}p_2(t)\delta_{II}C_{dd} + (C^2_{dw_g} - C_{dd}C_{w_gw_g})(p_2(t)\delta_{II} - f_{w_sw_g}) < 0$$

The hamiltonian respects the rule of the sign of the principal minor : $D_1 < 0$, $D_2 > 0$ and $D_3 < 0$. Therefore it is concave.

As it the sufficient conditions are defined with a lagrangian multiplier associated to each constraint, equation 54 differ from the dynamics of the co-state variable in the necessary condition.

$$\dot{p_1} - \rho p_1(t) = C_d - q_2(t) + q_3(t) \tag{70}$$

By knowing that these variable are associated with two opposite constraints, we can investigate whether these trajectories satisfy the sufficient conditions.

Lemma 8 *Over any interval of time* $\forall\, t \in]t_0, t_1[$
(i) such as $d(t) = 0$*, we cannot have an optimum. (ii) such as* $d(t) = \overline{d}$*, we cannot have an optimum.*

Proof 9 In a first step, notice that $\forall\ d = \{0; \overline{d}\}$*, we have* $\dot{d} = 0$ *and therefore* $R = w_g(t) + w_s(t)$*.*
The, by computing the time derivative of If the constraint $g_2(d(t))$ *is binding then* $q_2(t) = 0$ *and obviously* $q_3(t) > 0$*. Then, we can rewrite some conditions as follows :* $f_{w_g}(R) - C_{w_g}(w_g, d(t)) + p_1(t) = 0$ *with* $d(t) = \{0; \overline{d}\}$*, we get that* $\dot{p}_1 = \dot{w}_g C w_g w_g(w_g, 0) \geq 0$*.*
By using 54, we know that $\dot{p}_1 = C_{w_gw_g}(w_g, 0) + \frac{1}{\rho}C_d(w_g, 0) - f_{w_g}(R) - \frac{1}{\rho}q_2(t)$ *with* $d(t) = 0$*. By assumption 5, we deduce that this expression leads to a decreasing* $p_1(t)$*.*
To this end, we get two different expression for the dynamics of the co-state variable.
In a same way, we know that $\dot{p}_1 = C_{w_gw_g}(w_g, \overline{d}) + \frac{1}{\rho}C_d(w_g, \overline{d}) - f_{w_g}(R) + \frac{1}{\rho}q_3(t)$ *with* $d(t) = \overline{d}$*. If we assume that* $\dot{p}_1 = 0$ *then* $f_{w_g}(R) - C_{w_gw_g}(w_g, \overline{d}) - \frac{1}{\rho}C_d(w_g, \overline{d}) = \frac{1}{\rho}q_3(t)$*.*
However, $\lim_{w_g \to 0} f_{w_g}(R) - C_{w_gw_g}(w_g, \overline{d}) - \dfrac{1}{\rho}C_d(w_g, \overline{d}) < 0$ *that contradicts the fact that* $q_3(t) > 0$*.*

Proposition 4 *The interior solution satisfies the sufficient conditions and then, is the optimum.*

3.2 The interior solution

For the moment, we focus on the steady state. Given previous lemmas, we know that the steady state corresponds to the interior solution. First, rewrite the optimal condition.
From condition 44, we obtain :

$$p_1(t) = \partial_{w_g} C(w_g(t); d_t(t)) - \partial_{w_g} f(w_g(t) + w_s(t)) \tag{71}$$

Like the simple model, the shadow price of the resource is equal to the marginal production cost less the marginal productivity of the resource. Rearranging equation 48 yields :

$$\dot{p}_1 - \partial_{d_t} C(w_g; d_t) = \rho p_1(t) \tag{72}$$

Until now, there is no change.

From condition 46, we obtain :

$$p_2(t) = \Theta'(I(t)) \tag{73}$$

The optimal decision of investment means that the shadow price of capital is equal to the marginal adjustment cost. Firms invest to the point where the marginal value of capital equals its replacement cost. We observe as usual that the higher is $p_2(t)$, the larger is the investment.

$$I(t) = \Theta'^{-1}(p_2(t)) \Rightarrow \frac{dI(t)}{dp_2(t)} = \frac{1}{\Theta''(I(t))} > 0$$

Moreover, notice that $\Theta'(0) = 0 \Rightarrow \Theta'^{-1}(0) = 0$. Therefore, we can deduce that :

$$\begin{array}{l} I(t) > 0 \text{ if } p_2(t) > 0 \\ I(t) < 0 \text{ if } p_2(t) < 0 \end{array}$$

Rearranging equation 49 yields :

$$\dot{p}_2(t) + \partial_{w_g} C(w_g(t); d_t(t)) = (\rho + \partial_{w_s}\delta(w_s))\Theta'(I(t)) \tag{74}$$

The left-hand side is the sum of the forgone increase in value that would have been realized by investing a marginal unit at the next period and the marginal cost of extraction. The right-hand side is marginal cost of investment in term in monetary value realized after one period.

Taking the time derivative of equation 44 and using this result, 44 itself and 48 and Taking the time derivative of equation 46 and using this result, 46 itself and 49, we can reduce the dimensionality of the model and we obtain a system of four differential equations.

$$\begin{cases} \dot{w}_g(f_{w_g w_g} - C_{w_g w_g}) + (I(t) - \delta(w_s(t)))\, f_{w_g w_s} - (w_g + w_s - R)C_{dd} + C_d + \rho(C_{w_g} - f_{w_g}) \\ \dot{I}\Theta_{II} = [\rho + \delta_{w_s}]\Theta_I - C_{w_g} \\ \dot{d} = w_g + w_s - R \\ \dot{w}_s = I(t) - \delta(w_s(t)) \end{cases} \tag{75}$$

The next step consists of characterizing the steady state. It is obtained by setting the previous time derivatives to zero. By computing the system 75, we obtain reduce it to a system composed by two equations.

$$\begin{cases} \partial_{w_g} f(R) - \frac{1}{\rho}\partial_{d(t)} C(w_g; d(t)) = \partial_{w_g} C(w_g; d(t)) \\ \Theta[\delta(R - w_g)][\rho + \partial_{w_s}\delta(R - w_g)] = \partial_{w_g} C(w_g; d(t)) \end{cases} \tag{76}$$

We now investigate the equilibrium in the model. This task is helped by determining the signs of the slopes of both equations in the previous system. The slope is derived by taking differentials of the first equation with respect to w_g and d and solving for dd/dw_g to obtain :

$$\frac{dd}{dw_g} = -\frac{\partial^2_{w_g w_g} C(w_g; d(t)) + \frac{1}{\rho}\partial^2_{d(t) w_g} C(w_g; d(t))}{\partial^2_{w_g d} C(w_g; d(t)) + \frac{1}{\rho}\partial^2_{dd} C(w_g; d(t))} \tag{77}$$

The sign of 77 is straightforward. By assumption the cost function is strictly convex in $(w_g; d)$ therefore we know that its second derivatives are strictly positive. Therefore the slope is negative.
The slope of the second equation is derived by taking differentials with respect to w_g and d and solving for dd/dw_g. This gives the expression :

$$\frac{dd}{dw_g} = -\frac{\delta_{w_s}\Theta_{II}(\rho + \delta_{w_s}) + \delta_{w_s w_s}\Theta_I + C_{w_g w_g}}{C_{w_g d}} \tag{78}$$

The sign of 78 is straightforward. It is negative.

Given the signs of 77 and 78, it is necessary to verify the existence of a unique steady state.

Lemma 9 *There exists an optimum in the feasible region defined by the set $D = [0, R] \times [0, \overline{d}]$.*

Proof 10 Rename equation in the system such as :

$$\begin{cases} f_1(w_g(t), d(t)) = \partial_{w_g} f(R) - \frac{1}{\rho}\partial_{d(t)} C(w_g(t); d(t)) - \partial_{w_g} C(w_g(t); d(t)) = 0 \\ f_2(w_g(t), d(t)) = \Theta[\delta(R - w_g)][\rho + \partial_{w_s}\delta(R - w_g)] - \partial_{w_g} C(w_g; d(t)) = 0 \end{cases}$$

Now, we verify that :

$$\begin{aligned}
\lim_{(w_g,d)\to(R,0)} f_1(w_g(t), d(t)) &= \partial_{w_g} f(R) - \frac{1}{\rho}\partial_{d(t)} C(R; 0) - \partial_{w_g} C(R; 0) > 0 \textit{ with } 5 \\
\lim_{(w_g,d)\to(0,\overline{d})} f_1(w_g(t), d(t)) &= \partial_{w_g} f(R) - \frac{1}{\rho}\partial_{d(t)} C(0; \overline{d}) - \partial_{w_g} C(0; \overline{d}) < 0 \textit{ with } 6 \\
\lim_{(w_g,d)\to(R,0)} f_2(w_g(t), d(t)) &= -\partial_{w_g} C(R; 0) < 0 \\
\lim_{(w_g,d)\to(0,\overline{d})} f_2(w_g(t), d(t)) &= \Theta[\delta(R)][\rho + \partial_{w_s}\delta(R)] - \partial_{w_g} C(0; \overline{d}) > 0 \textit{ with}
\end{aligned}$$

Lemma 10 *The optimum is unique.*

Proof 11 Letting J_{mn} denote an element in row m and column n of the matrix, the system of equations 75 implies the following elements : $J_{11} = -C_{w_g w_g} - \frac{1}{\rho}C_{w_g d}$, $J_{12} = -C_{w_g d} - \frac{1}{\rho}C_{dd}$, $J_{21} = Z - C_{w_g w_g}$, $J_{22} = -C_{w_g d}$ with $Z = -\delta_{w_s}\Theta_{II}(\rho + \delta_{w_s}) - \delta_{w_s w_s}\Theta_I$. Note that all the elements are negative therefore the sign of the determinant is not clear. However we can rewrite the matrix as a sum of three matrix without changing

the determinant.
Notice that

$$\begin{pmatrix} 0+J_{11} & 0+J_{12} \\ Z+\widehat{J}_{21} & 0+J_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ \widehat{J}_{21} & J_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ Z & J_{22} \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ Z & 0 \end{pmatrix}$$

with $\widehat{J}_{21} = -C_{w_g w_g}$.
If we calculate the determinant, we have : $J_{11}J_{22} - J_{12}\widehat{J}_{21} - J_{12}Z$. *Namely* $J_{11}J_{22} - J_{12}\widehat{J}_{21} = \frac{1}{\rho}\left((C_{w_g d})^2 - C_{w_g w_g}C_{dd}\right) < 0$ *and we know that* $Z < 0$ *and* $J_{12} < 0$. *Therefore, we can deduce that the determinant of the jacobian matrix is negative.* ■

Both following figures represent the steady state with investment.

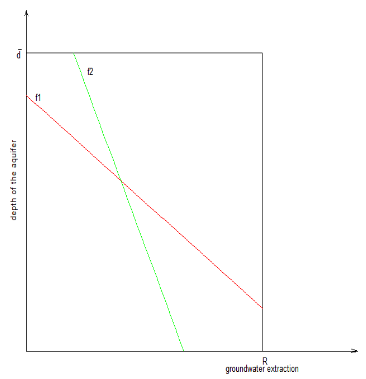

Figure 2: Illustration of of unique solution

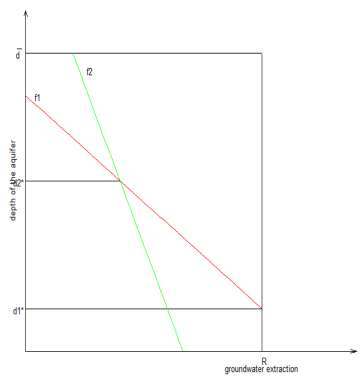

Figure 3: Comparison of both regimes

Proposition 5 *In the model with investment, the depth of the aquifer is higher than in the model without investment possibility. Moreover, the amount of withdrawn resource is smaller in the model with investment than in the simple model.*

Proof 12 $\exists d \in]0, \bar{d}[$, $\lim_{w_g \to R} f_1(w_g(t), d(t)) = 0 f'(R) - \frac{1}{\rho}\partial_{d(t)}C(R; d(t)) - \partial_{w_g}C(R; d(t)) = 0$ $d(t) = d^*(t)$ *with* $d^*(t)$ *corresponding to the depth of the aquifer that characterizes the steady state in the simple model 31.*
Given the proof of the existence of the equilibrium, we know that solution on the boundaries is impossible and with the definition of the slope 77, if $w_g(t)$ *decreases then the depth* $d(t)$ *increases.*
Then, the optimal depth in the model with investment is higher than in the first model.

This proposition outlines an interesting results. There is a negative effect on the level of the water table. By investing we prevent the total amount of the recharge to reach the aquifer. This fact is quite straightforward because as soon as a farmer invests in capital to collect rainwater, he cuts a part of the natural recharge that will reach the aquifer. However, the level of extraction is also reduced and correspond to the amount that replenishes the aquifer. Because of the substitution of both inputs, we always produce at the recharge but it seems that in the share of the recharge between both reservoirs, some amount are lost.

Conclusion and policy implications

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Appendix

$w_s(t)$	$w_g(t)$	d(t)	$\overline{d} - d(t)$
0	0	0	0
0	+	0	0
+	0	0	0
+	+	0	0
0	0	0	+
0	0	+	0
0	0	+	+
0	+	+	0
0	+	0	+
0	+	+	+
+	0	+	0
+	0	0	+
+	0	+	+
+	+	+	0
+	+	0	+
+	+	+	+