Pricing levered warrants with dilution using observable variables*

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This version: March 7, 2009

Abstract

We propose a valuation framework for pricing European call warrants on the issuer's own stock. We allow for debt in the issuer firm. In contrast to other works which also price warrants with dilution issued by levered firms, ours uses only observable variables. Thus, we extend the models of both Crouhy and Galai (1994) and Uhkov (2004). We provide numerical examples to study some implementation issues and to compare the model with existing ones.

Journal of Economic Literature classification: C13, C63, G32.

Keywords: Warrant pricing, dilution, leverage, observable variables.

*We are grateful for valuable comments from David Abad, Louis Ederington, Stewart Hodges, Belén Nieto, Gonzalo Rubio, Andrey Ukhov and seminar participants at the XLII Conference of the Euro Working Group in Financial Modelling and Cardenal Herrera-CEU University. The authors acknowledge the financial support from the Spanish Ministry of Education and Science, Junta de Andalucía and Gobierno de Navarra (grants SEJ2006-14809-C03-01, ECO2008-03058, P06-SEJ-01537 and P08-SEJ-03917). The usual caveat applies.

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1 Introduction

As European call options, European call warrants give the holder the right to purchase a specified amount of an asset at an agreed price, on a fixed date. There are two types of warrants: warrants on the company's own stock and warrants on other assets. In the former case, the exercise of the warrant in exchange for new shares results in a dilution of the firm's own stock. To allow for possible dilution when pricing warrants, some studies, such as Galai and Schneller (1978), Noreen and Wolfson (1981), Galai (1989), and Lauterbach and Schultz (1990), present different revisions of the Black and Scholes (1973) option pricing model. In the valuation formulas obtained by these studies, firm market value and its volatility need to be known, which is not possible. Moreover, when there are warrants outstanding, the firm value is itself a function of the warrant price.

To overcome these problems, Schulz and Trautmann (1994) propose a warrant-pricing procedure based on the price and volatility of the underlying stock, both of which are observable variables. More recently, Ukhov (2004) develops an algorithm that generalizes the Schulz and Trautmann (1994) proposal for the case of the warrant ratio[1] being distinct from unity.

The above studies value warrants issued by companies financed by shares and warrants. The majority of firms, however, are also debt financed. To reflect this fact, Crouhy and Galai (1994) develop a pricing model for the valuation of warrants issued by levered companies. Later, Koziol (2006) extends the analysis of Crouhy and Galai to explore optimal warrant exercise strategies in the case of American-type warrants.

Both the Crouhy-Galai's formula and its extension in Koziol (2006) depend on the value of a firm with the same investment policy as the one issuing the warrant but financed entirely with shares of stock. Therefore, these pricing models again present the drawback of dependence on unobservable variables. In this paper, we devise a model for the valuation of warrants issued by levered companies, where only the values of observable variables need to be known.

The remainder of the study is organized as follows. Section 2 briefly describes the va-

[1]We use the term ratio to refer to the number of units of the underlying asset that can be purchased by exercising a call warrant.

luation of unlevered warrants with dilution. Section 3 presents a valuation framework for pricing warrants on own stock issued by debt-financed firms that uses variables that can be observed. Section 4 examines its implementation through some numerical examples. Finally, section 5 contains the conclusions of our research.

2 Pricing unlevered warrants with dilution

A recurring issue in the corporate warrant pricing literature is the fact that the value of a warrant is a function of firm value, which in turn includes the warrant value and is unobservable. Authors such as Ingersoll (1987), Galai (1989), Crouhy and Galai (1991) and Veld (2003) explicitly acknowledge this problem, and provide different alternatives. More recently, Ukhov (2004) draws on the work of Schulz and Trautmann (1987) and proposes an algorithm that requires only knowledge of observable variables. First, he follows the work of Ingersoll (1987) and derive an expression of warrant value as a function of firm value and return volatility, then he establishes a relationship between these variables and the price and volatility of the underlying stock. In this section, we introduce a unifying notation and we briefly present the models of Ingersoll (1987) and Ukhov (2004).

2.1 Valuation of unlevered warrants using unobservable variables

Let there be a firm financed by N shares of stock and M European call warrants. Each warrant gives the holder the right to k shares at time $t = T$ in exchange for the payment of an amount X. Let V_t be the asset value of the firm at time t, S_t and σ_S are the price and volatility of the underlying share, respectively, and let w_t be the warrant price at time t.

If the M warrants are exercised at $t = T$, the firm receives an amount of money MX and issues Mk new shares of stock. Thus, immediately before the exercise of the warrants, each warrant must be worth $\frac{k}{N+kM}(V_T + MX) - X$. According to Ingersoll (1987), warrant holders will exercise the warrants only when this value is non-negative, that is, when $kV_T \geq NX$. Thus, the warrant price at date of exercise can be expressed as follows:

$$w_T = \frac{1}{N+kM} \max\left(kV_T - NX, 0\right) \tag{1}$$

Assuming that the assumpyions of Black and Scholes (1973) hold, Ingersoll obtains the following expression for the warrant price:

$$w_t = \frac{1}{N+kM}\left[kV_t\Phi(d_1) - e^{-r(T-t)}NX\Phi(d_2)\right] \tag{2}$$

with:

$$d_1 = \frac{\ln(kV_t/NX) + (r+\sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}} \tag{3}$$

$$d_2 = d_1 - \sigma_V\sqrt{T-t} \tag{4}$$

where $\Phi(\cdot)$ is the distribution function of a Normal random variable and σ_V is the return volatility of V_t.

As we can see, the warrant pricing formula proposed by Ingersoll (1987) depends on V_t y σ_V, which are unobservable values.

2.2 Valuation of unlevered warrants using observable variables

To obtain a warrant-pricing formula where only the values of observable variables need to be known, Ukhov (2004) draws on expressions (2) - (4) and proposes relating V_t and σ_V to the underlying share price, S_t, and its return volatility, σ_S. He relates these variables as follows:

$$\sigma_S = \sigma_V\Delta_S\frac{V_t}{S_t} \tag{5}$$

where $\Delta_S = \partial S_t/\partial V_t$. Given that $V_t = NS_t + Mw_t$, the following expression is satisfied:

$$N\Delta_S + M\Delta_w = \Delta_V = 1 \tag{6}$$

where $\Delta_w = \partial w(V_t;\cdot)/\partial V_t$. Furthermore, using (2) we have that:

$$\Delta_w = \frac{k}{N+kM}\Phi(d_1) \tag{7}$$

Substituting the above into (6), the expression for Δ_S is obtained:

$$\Delta_S = \frac{1-M\Delta_w}{N} = \frac{N+kM-kM\Phi(d_1)}{N(N+kM)} \tag{8}$$

Finally, substituting the expression (8) into (5) the relationship between the unobservable variables, V_t and σ_V, and the observable variables S_t and σ_S is given.

Having established this relationship, Ukhov (2004) proposes the following algorithm to obtain the warrant price:

1. Solve (numerically) the following system of non-linear equations for (V_t^*, σ_V^*):

$$\begin{cases} NS_t = V_t - \frac{M}{N+kM}\left[kV_t\Phi(d_1) - e^{-r(T-t)}NX\Phi(d_2)\right] \\ \sigma_S = \frac{V_t}{S_t}\Delta_S\sigma_V \end{cases} \tag{9}$$

   with:

$$\Delta_S = \frac{N+kM-kM\Phi(d_1)}{N(N+kM)} \tag{10}$$

   and where:

$$d_1 = \frac{\ln(kV_t/NX)+(r+\sigma_V^2/2)(T-t)}{\sigma_V\sqrt{T-t}} \tag{11}$$

$$d_2 = d_1 - \sigma_V\sqrt{T-t} \tag{12}$$

2. The warrant price, w_t, is computed as:

$$w_t = \frac{V_t^* - NS_t}{M} \tag{13}$$

This way Ukhov provides a valuation formula for the warrant price based on observable variables.

3 Pricing levered warrants with dilution

Despite the advantage of using only the values of observable variables, the Ukhov (2004) model has the limitation of assuming that the issuer of the warrant is a pure-equity firm, since the majority of firms issuing warrants are also debt financed.

In this section, we extend the work of Ukhov allowing for debt in the issuer firm. As we shall see, this extension is by no means obvious. Specifically, we consider a firm financed by N shares of stock, M European call warrants and debt D. The debt consists of a zero-coupon bond with face value F and maturity T_D. For every warrant held, the holder has the right to purchase k shares of stock at T, in exchange for the payment of an amount X.

As other authors (see Ingersoll, 1987 and Crouhy and Galai, 1994 among others), we assume that the proceeds from exercising the warrants are reinvested in the company. Similarly, we also assume no economies of scale and a stationary return distribution for one unit of investment, independent of firm size. Due to this assumption, in case of the exercise of warrants, the value of the company increases and the number of shares outstanding also changes. This fact has a different effect on the price of the warrant depending on if the firm debt has matured previously, or it is still alive. Accordingly, we consider three cases: a) warrants expire before debt ($T < T_D$); b) warrants have the same maturity as debt ($T = T_D$); and c) warrants expire after the zero coupon bond ($T > T_D$).

To obtain the pricing formula in each case, we follow Ukhov (2004) and express the value of the levered warrant as a function of the unobservable variables. Then, we establish a relationship between the unobservable variables and the underlying stock price and its return volatility.

3.1 Warrants with shorter maturity than debt

Let us consider the case in which the warrant issuer is financed with a zero-coupon bond with longer maturity than the exercise date of the warrants, that is, $T < T_D$. We first build on Crouhy and Galai (1994) and obtain an expression for the value of the warrants that depends on unobservable variables.

Crouhy and Galai (1994) propose a pricing formula for levered warrants when debt maturity is longer than the exercise date of warrants. In their formula, the warrant price depends on the value of a firm with the same investment policy as the firm issuing the warrant, but financed entirely by common stock. Thus, the initial value of the reference firm is the same as the one of the levered firm. The assumptions from which Crouhy and Galai (1994) derive their results are that the risk-free interest rate, r, is known and constant, and perfect market conditions.

Let us suppose that at $t = 0$ the reference firm issues N' shares of stock at a price

$V_0'/N' = S_0'$, while the warrant-issuing firm issues N shares of stock, M warrants and a zero-coupon bond with maturity $T_D > T$. Thus we have that for $0 \leq t < T$:

$$V_t = NS_t + Mw_t + D_t, \text{ with } V_t = V_t' \tag{14}$$

where S_t, w_t and D_t are the value of a share, a warrant and the debt of the levered firm at time t. Thus, the warrant value at any time prior to the exercise date is given by the following expression:

$$w_t = \frac{V_t' - NS_t - D_t}{M}, \text{ with } t < T \tag{15}$$

As Crouhy and Galai (1994), we begin by analyzing the value of the company at the maturity date of debt ($t = T_D$). If the warrants are exercised at $t = T$ an amount MX is reinvested in the company, thus, the value of the levered company as of the date of exercise may differ from the reference firm value. If the warrants have not been exercised at $t = T$, the value of the levered company at $t = T_D$ will be be equal to the reference asset value, V_{T_D}', whereas if the warrants have been exercised at $t = T$, the value of the levered company at $t = T_D$ will be $V_{T_D}'(1 + MX/V_T')$, where V_T' is the reference asset value at $t = T$. The ratio $\frac{MX}{V_T'}$ measures the expansion of the company's assets at $t = T$.

The exercise of the warrants at $t = T$ depends on whether the value of the shares received by the warrant-holders is greater than the exercise price. Although the traditional analysis[2] considers that warrants should be exercised if the value of the shares immediately prior to the exercise date is greater than X, Crouhy and Galai (1994) show that this condition may lead to erroneous decisions and argue that warrants should be exercised if the value of the shares of stock is greater than X immediately after the expiration.

As mentioned earlier, we assume that each warrant gives the holder the right to buy k shares of stock [3], with $k > 0$. Thus, we can write the post-expiration value of a share of stock at $t = T$, S_T, as follows:

$$S_T = \begin{cases} \frac{V_T' - D_T^{NW}}{N} \equiv S_T^{NW} & \text{if warrants are not exercised at } t = T \\ \frac{V_T' + MX - D_T^{W}}{N + kM} \equiv S_T^{W} & \text{if warrants are exercised at } t = T \end{cases} \tag{16}$$

[2]See for example Ingersoll (1987), Schulz and Trautmann (1994) and Ukhov (2004).

[3]We should note that in their work, Crouhy and Galai (1994) only consider the case in which $k = 1$, that is, each warrant entitles the right to purchase one share of stock.

where V_T' is the reference firm value at $t = T$, and D_T^W, D_T^{NW}, S_T^W and S_T^{NW} denote the debt value and the price of a share of stock in the company immediately after T with warrants exercised and without warrants exercised, respectively. Given that S_T^W is an increasing function of V_T', there exists a unique value of V_T', $\bar{V}_T'$, for which the warrant-holders are indifferent as to whether to exercise their warrants or let them expire, that is, $kS_T^W(\bar{V}_T') \equiv X$. Thus, for reference asset values above (below) $\bar{V}_T'$, the warrants will (will not) be exercised at $t = T$.

Alternatively, we can write the above expression in the following way:

$$S_T = \begin{cases} \frac{c(V_T',F,T_D-T)}{N} \equiv S_T^{NW} & \text{if } V_T' \leq \bar{V}_T' \\ \frac{c(V_T'+MX,F,T_D-T)}{N+kM} \equiv S_T^W & \text{if } V_T' > \bar{V}_T' \end{cases} \tag{17}$$

where $c(x, K, T)$ denotes the value of a European call option on x, with strike K and time to maturity T, and where $\bar{V}_T'$ is the reference firm value at which the warrants may be exercised. Consequently, at any time t, with $T < t \leq T_D$, the value of one share of stock may be expressed as follows:

$$S_t = \begin{cases} \frac{c(V_t',F,T_D-t)}{N} \equiv S_t^{NW} & \text{if } V_T' \leq \bar{V}_T' \\ \frac{c(V_t'+MX,F,T_D-t)}{N+kM} \equiv S_t^W & \text{if } V_T' > \bar{V}_T' \end{cases} \tag{18}$$

With the assumptions that the reference asset value V_t' follows a lognormal process and that there are no arbitrage opportunities, there exists a risk-neutral probability measure under which $e^{-rt}V_t'$ is a martingale, so that we can write:

$$dV_t' = rV_t'dt + \sigma_{V'}V_t'dZ_t' \tag{19}$$

where r is the risk-free interest rate, $\sigma_{V'}$ is the return volatility of V_t', and Z_t' is a standard Brownian motion. Therefore, we can apply the Black and Scholes (1973) option pricing formula to the systems (17) and (18) and thus obtain the value of S_t, with $T \leq t \leq T_D$.

A consequence of the above assumption is that for any time t, with $t < T$, we can value the firm's shares discounting their expected value at T at the risk-free discount rate, r:

$$\begin{aligned} S_t &= e^{-r(T-t)}E^*[S_T] \\ &= e^{-r(T-t)}E^*\left[\frac{c(V_T',F,T_D-T)}{N}I_{V_T' \leq \bar{V}_T'} + \frac{c(V_T'+MX,F,T_D-T)}{N+kM}I_{V_T' > \bar{V}_T'}|\mathscr{F}_t\right] \end{aligned} \tag{20}$$

where E^* denotes the expected value under the risk-free probability measure, $\mathscr{F}_t$ is the available information set at time t, and $I_{[condition]}$ is an indicator that takes a value of 1 when the condition is satisfied and 0 otherwise.

We know that the solution of the process given by (19) is:

$$V'_T = V'_t \exp\left((r - 1/2\sigma^2_{V'})(T-t) + \sigma_{V'}(Z'_T - Z'_t)\right) \tag{21}$$

Thus, V'_T follows a lognormal distribution, that is, $[\ln V'_T]|V'_t \sim \Phi\big(\ln V'_t + (r - 0.5\sigma^2_{V'})(T - t), \sigma^2_{V'}(T-t)\big)$.

From the properties of the lognormal distribution, expression (20) can be rewritten as follows:

$$S_t = e^{-r(T-t)}\left(\int_0^{\bar{V}'_T} \frac{c(V'_T, F, T_D - T)}{N} f(V'_T) dV'_T + \int_{\bar{V}'_T}^{\infty} \frac{c(V'_T + MX, F, T_D - T)}{N + kM} f(V'_T) dV'_T\right) \tag{22}$$

where $f(\cdot)$ is the probability distribution function of a lognormal random variable. Finally, defining $y(V'_T) = \frac{\ln \frac{V'_T}{V'_t} + (r - \frac{1}{2}\sigma^2_{V'})(T-t)}{\sigma_{V'}\sqrt{T-t}}$, we can compute the stock price as follows:

$$S_t = \frac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}}\left(\int_{-\infty}^{\bar{y}} \frac{c(V'_T, F, T_D - T)}{N} e^{-\frac{y^2}{2}} dy + \int_{\bar{y}}^{\infty} \frac{c(V'_T + MX, F, T_D - T)}{N + kM} e^{-\frac{y^2}{2}} dy\right) \tag{23}$$

Analogously, the value of debt at time t, with $t < T$, is given by:

$$\begin{aligned} D_t &= e^{-r(T-t)} E^*[D_T] \\ &= e^{-r(T-t)} E^*\Big[(Fe^{-r(T_D-T)} - p(V'_T, F, T_D - T)) I_{V'_T \le \bar{V}'_T} \\ &+ (Fe^{-r(T_D-T)} - p(V'_T + MX, F, T_D - T)) I_{V'_T > \bar{V}'_T} | \mathscr{F}_t\Big] \end{aligned} \tag{24}$$

where $p(x, K, T)$ denotes the value of a European put option on x, with strike K and time to maturity T. Using the same reasoning as for the share value, we obtain the following:

$$\begin{aligned} D_t &= Fe^{-r(T_D-t)} \\ &- \frac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}}\left(\int_{-\infty}^{\bar{y}} p(V'_T, F, T_D - T) e^{-\frac{y^2}{2}} dy + \int_{\bar{y}}^{\infty} p(V'_T + MX, F, T_D - T) e^{-\frac{y^2}{2}} dy\right) \end{aligned} \tag{25}$$

Once we have the expressions for S_t and D_t, we substitute them into equation (15) to obtain the warrant price, w_t, as a function of the reference asset value and its return volatility, V_t' and $\sigma_{V'}$. It should be noted that, for $t < T$, both the reference firm value and its return volatility are equal to those of the levered firm, that is, $V_t' = V_t$ and $\sigma_{V'} = \sigma_V$. Thus, following the Crouhy and Galai approach we have obtained an expression for w_t, with $t < T$, that depends on the levered firm value and its return volatility, V_t and σ_V.

Once we have the price of the warrant expressed as a function of the reference asset value and return volatility, V_t' and $\sigma_{V'}$, based on Ukhov (2004) we propose to establish a relationship between these variables and the firm's stock price, S_t, and its return volatility, σ_S. To relate these variables, we use the expression (23), which relates the variables V_t' and $\sigma_{V'}$ to the stock price, S_t, and also the following expression to relate V_t', $\sigma_{V'}$ and S_t to σ_S:

$$\sigma_S = \sigma_{V'} \frac{\partial S_t}{\partial V_t'} \frac{V_t'}{S_t} \tag{26}$$

where S_t is given by (23).

Having related the unobservable and observable variables, we formulate the following proposition:

Proposition 1 *Let us consider a company with value denoted by V_t, and financed by N shares of stock, M European corporate call warrants with exercise date T, and a zero-coupon bond with face value F and maturity T_D, with $T_D > T$. For every warrant held, the warrant holder has the right to k shares in the company in exchange for payment of an amount X at time $t = T$. Let S_t be the stock price and let σ_S be the stock return volatility. Let V_t' be the value of a firm with the same investment policy but financed entirely by shares. The value of this firm and its return volatility are equal to the value of the levered firm and its volatility for $t < T_D$ if the warrants are not exercised at $t = T$, and for $t < T$ if the warrants are exercised at $t = T$. Furthermore, if V_t' follows a geometric Brownian motion with standard deviation $\sigma_{V'}$ under a risk-neutral probability measure and in the absence of arbitrage opportunities, then the value at time t of a European call warrant on the company's shares will be given by the following algorithm:*

1. *Solve (numerically) the following system of non-linear equations for* $(V_t'^*, \sigma_{V'}^*)$*:*

$$\begin{cases} S_t = \frac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}}\left(\int_{-\infty}^{\bar{y}} \frac{c(V_T',F,T_D-T)}{N} e^{-\frac{y^2}{2}}dy + \int_{\bar{y}}^{\infty} \frac{c(V_T'+MX,F,T_D-T)}{N+kM} e^{-\frac{y^2}{2}}dy\right) \\ \sigma_S = \sigma_{V'} \frac{\partial S_t}{\partial V_t'} \frac{V_t'}{S_t} \end{cases} \tag{27}$$

where $c(x,K,T)$ *denotes the value of a European call option on x, with strike K and time to maturity* T*, whereas* $\bar{V}_T'$ *denotes the value of* V_T' *that satisfies* $k\frac{c(V_T'+MX,F,T_D-T)}{N+kM} = X$*,* $\bar{y} = y(\bar{V}_T')$*, and* $y(V_T') = \frac{\ln\frac{V_T'}{V_t'}+(r-\frac{1}{2}\sigma_{V'}^2)(T-t)}{\sigma_{V'}\sqrt{T-t}}$.

2. *The warrant price at time t, with* $t < T$*, is obtained as:*

$$w_t = \frac{V_t'^* - NS_t - D_t}{M} \tag{28}$$

where D_t *is given by:*

$$D_t = Fe^{-r(T_D-t)}$$

$$-\frac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}}\left(\int_{-\infty}^{\bar{y}} p(V_T'^*,F,T_D-T)e^{-\frac{y^2}{2}}dy + \int_{\bar{y}}^{\infty} p(V_T'^*+MX,F,T_D-T)e^{-\frac{y^2}{2}}dy\right) \tag{29}$$

and where $p(x,K,T)$ *is the value of a European put option on x, with strike price K and time to maturity* T.

It should be noted that our proposed algorithm is based on observable variables only, such as the risk-free interest rate and the current price of the underlying stock. Thus, we can claim to have solved a problem found in the literature concerning the valuation of corporate warrants issued by levered firms when $T < T_D$.

3.2 Warrants with the same maturity as debt

Let us suppose now that the warrants issued by the company have the same maturity as debt, that is $T = T_D$. This case is consistent with many issues of warrants that are joint to some bond issues. In this situation, the owner of a warrant has the right to pay X at T and receive k shares of stock with individual value $\frac{1}{N+kM}(E_T + MX)$, where E_T is the value of equity at T, just after the maturity of debt. We can thus express the value of the warrant at $t = T$ as:

$$w_T = \max(0,\ k\lambda(E_T + MX) - X) \tag{30}$$

where $\lambda = \frac{1}{N+kM}$. Furthermore, we know that $E_T = \max(V_T - F, 0)$, because if the value of the company at T is larger than the face value of debt, F, debtholders get F while shareholders get $V_T - F$, and in case of default, the debtholders receive what is left of the company, V_T, while the shareholders get 0. Thus, we can write (30) this way:

$$w_T = \max\Big(0,\ \max\big(k\lambda(V_T - F + MX) - X, -\lambda NX\big)\Big) \tag{31}$$

Additionally, since the values of λ, N and X are non-negative, we can express w_T as follows:

$$w_T = \lambda \max(0, kV_T - kF - NX) \tag{32}$$

We must note that at time $t = T$ the warrantholder receives the same payoff as the owner of λ European call options on kV_t, with strike $kF + NX$ and exercise date at T. Thus, if we assume that the Black-Scholes assumptions are satisfied, the value of the warrant is given by the following expression:

$$w(V_t,\ \sigma,\ X) = \lambda\left[kV_t\Phi(f_1) - e^{-r(T-t)}(kF + NX)\Phi(f_2)\right] \tag{33}$$

with:

$$f_1 = \frac{\ln\left(\frac{kV_t}{kF+NX}\right) + \left(r + \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}} \tag{34}$$

$$f_2 = f_1 - \sigma_V\sqrt{T-t} \tag{35}$$

where $\Phi(\cdot)$ is the distribution function of a Normal random variable and where σ_V is the standard deviation of V_t.

This way we have expressed the value of the warrant as a function of the firm value, V_t, and its volatility, σ. Since these variables cannot be observed, on the basis of Ukhov (2004) we search for a relationship between V_t and σ_V with S_t and σ_S. As we have seen before, we can establish a relationship by this expression:

$$\sigma_S = \frac{V_t}{S_t}\Delta_S\sigma_V \tag{36}$$

where $\Delta_S = \frac{\partial S_t}{\partial V_t}$. To compute Δ_S when there exists debt we see that now $V_t = NS_t + Mw_t + D_t$, so we have that:

$$\Delta_V = 1 = N\Delta_S + M\Delta_w + \Delta_D \tag{37}$$

Using (33) we obtain the following:

$$\Delta_w = \frac{\partial w_t}{\partial V_t} = k\lambda\Phi(f_1) \tag{38}$$

On the other hand, to obtain the expression for Δ_D first we must determine the expression for D_t. We know that the payoff received by debtholders at maturity can be written this way: $D_T = \min(F,\ V_T) = F - \max(0,\ F - V_T)$. Thus, D_t can be expressed as:

$$D_t = Fe^{-r(T-t)} - p(V_t, F, T-t) \tag{39}$$

where $p(x,K,T)$ is the value of a European put option on x with strike price K and time to maturity T. Thus, Δ_D is given by this expression:

$$\Delta_D = \frac{\partial D_t}{\partial V_t} = 1 - \Phi(h_1) \tag{40}$$

where:

$$h_1 = \frac{\ln\frac{V_t}{F} + \left(r + \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}} \tag{41}$$

Once we know the expressions for Δ_w and Δ_D and substituting in (37), we obtain the expression for Δ_S and therefore, we have V_t related to σ_V, S_t and σ_S when the firm is financed by equity, warrants and debt and $T = T_D$.

Furthermore, we can consider that stockholders and warrantholders have a European call option on the value of the firm, with exercise price equal to the face value of the debt, and with maturity at T, that is, $NS_t + Mw_t = c(V_t,\ F,\ T-t)$. Moreover, using the put-call parity we can check that $V_t = NS_t + Mw_t + D_t$ is satisfied for all $t \in [0,\ T]$.

Having established the relationship between the unobservable and observable variables, we can now enunciate our second proposition:

Proposition 2 *Let us consider a company with value denoted by V_t, and financed by N shares of stock, M European corporate call warrants with exercise date T, and a zero-coupon bond with face value F and maturity T. For every warrant held, the warrant* *holder has the right to k shares in the company in exchange for payment of an amount X at time $t = T$. Let S_t be the share price and let σ_S be the share return volatility. If V_t follows a geometric Brownian motion with standard deviation σ_V, then the value at time t of a European call warrant on the company's shares will be given by the following algorithm:*

1. *Solve (numerically) the following system of nonlinear equations for (V_t^*, σ^*):*

$$\begin{cases} N\,S_t = V_t\Phi(h_1) - e^{-r(T-t)}F\Phi(h_2) - M\lambda\left[kV_t\Phi(f_1) - e^{-r(T-t)}(kF+NX)\Phi(f_2)\right] \\ \sigma_S = \frac{V_t}{S_t}\Delta_S\sigma_V \end{cases} \tag{42}$$

with:

$$\Delta_S = \frac{\Phi(h_1) - \frac{kM}{N+kM}\Phi(f_1)}{N} \tag{43}$$

$$f_1 = \frac{\ln\left(\frac{kV_t}{kF+NX}\right) + \left(r + \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}} \tag{44}$$

$$f_2 = f_1 - \sigma_V\sqrt{T-t} \tag{45}$$

$$h_1 = \frac{\ln\left(\frac{V_t}{F}\right) + \left(r + \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{46}$$

$$h_2 = h_1 - \sigma_V\sqrt{T-t} \tag{47}$$

and where $\lambda = \frac{1}{N+kM}$.

2. *The warrant price at t is obtained as:*

$$w_t = \lambda\left[kV_t^*\Phi(f_1) - e^{-r(T-t)}(kF+NX)\Phi(f_2)\right] \tag{48}$$

We must remark that the formula obtained represents an extension of Ukhov's model to consider the possibility of the firm financed with debt. Moreover, when the firm has no debt, we can verify that the expression we obtain is the same as that given by Ukhov (2004). Furthermore, if the firm has no debt and the effect of dilution is minimal, that is, $\frac{M}{N} \to 0$, the pricing formula collapses to the Black-Scholes model.

3.3 Warrants with longer maturity than debt

Let us consider now that case of warrants with longer maturity than debt ($T > T_D$). Thus, at $t = T$ the owner of a warrant has the right to pay X and receive k shares of stock with individual value $\frac{V_T}{N+kM}$, where V_T is the value of the company at T.

In the same way as Crouhy and Galai (1994) we are going to express V_T as a function of the value of a reference firm with the same investment policy as the warrant issuer but financed only with shares of stock. For any time prior to the maturity of debt, $t < T_D$, it is satisfied that the values of the two companies are the same, that is:

$$V_t = NS_t + Mw_t + D_t, \text{ with } V_t = V_t' \tag{49}$$

where V_t is the value of the issuer company, S_t is the value of an individual stock of the company, w_t is the value of a warrant, D_t is the value of debt, and V_t' denotes the value of the reference firm.

Moreover, we know that at $t = T_D$, if the value of the issuer firm is larger than the face value of debt, F, debholders get F while shareholders get the rest of the firm value, and if the contrary, the firm defaults and debtholders receive what is left of the company, while shareholders get 0. In terms of the value of the reference company, we can express the value of the issuer's firm this way:

$$V_{T_D} = \begin{cases} 0 & \text{if } V'_{T_D} < F \\ V'_{T_D} - F & \text{if } V'_{T_D} \geq F \end{cases} \tag{50}$$

And at $t = T$, just after the expiration date of the warrants, we can express V_T as this:

$$V_T = \begin{cases} 0 & \text{if } V'_{T_D} < F \\ V'_T - F & \text{if } V'_{T_D} \geq F \text{ and the warrants are not exercised at } t = T \\ V'_T - F + MX & \text{if } V'_{T_D} \geq F \text{ and the warrants are exercised at } t = T \end{cases} \tag{51}$$

The condition for a warrantholder to exercise a warrant at $t = T$ is that the value of the k shares of stock he or she would receive in case of exercise be greater than the strike price, that is, $k\frac{V'_T - F + MX}{N+kM} \geq X$. This way, we can write the value at $t = T$ of a warrant as:

$$w_T = \begin{cases} 0 & \text{if } V'_{T_D} < F \\ \lambda \max(0, kV'_T - kF - NX) & \text{if } V'_{T_D} \geq F \end{cases} \tag{52}$$

where $\lambda = \frac{1}{N+kM}$. Consequently, we can consider that at $t = T_D$, just after maturity of debt, the warrant value is:

$$w_{T_D} = \begin{cases} 0 & \text{if } V'_{T_D} < F \\ \lambda c(kV'_{T_D}, kF + NX, T - T_D) & \text{if } V'_{T_D} \geq F \end{cases} \tag{53}$$

where $c(x, K, T)$ denotes the value of a European call option on x, with strike K and time to maturity T.

As in the case of warrants with shorter maturity than debt, if we suppose that the value of the reference firm follows a lognormal process and the absence of arbitrage opportunities, then it is satisfied that there exists a risk-neutral probability measure below which $e^{-rt}V'_t$ is a martingale, such that equation (19) is satisfied. As a consequence, we can value the warrant discounting its expected value at T_D at the risk-free discount rate, r, that is:

$$w_t = e^{-r(T_D-t)}E^*[w_{T_D}] = e^{-r(T_D-t)}E^*\left[\lambda c(kV'_{T_D}, kF + NX, T - T_D)I_{V'_{T_D} \geq F}|\mathscr{F}_t\right] \tag{54}$$

where E^* denotes the expected value under the risk-free probability measure, $\mathscr{F}_t$ is the available information set at time t and $I_{[condition]}$ is an indicator that takes a value of 1 when the condition is satisfied and 0 otherwise. Using the same reasoning as in subsection 3.1. we can write w_t this way:

$$w_t = \frac{e^{-r(T_D-t)}}{\sqrt{2\pi(T_D - t)}} \int_{\bar{y}}^{\infty} \lambda c(V'_{T_D}, F, T - T_D) e^{-\frac{y^2}{2}} dy \tag{55}$$

with $y(V'_{T_D}) = \frac{\ln \frac{V'_{T_D}}{V'_t} + (r - \frac{1}{2}\sigma^2_{V'})(T_D - t)}{\sigma_{V'}\sqrt{T_D - t}}$, $\bar{y} = \frac{\ln \frac{F}{V'_t} + (r - \frac{1}{2}\sigma^2_{V'})(T_D - t)}{\sigma_{V'}\sqrt{T_D - t}}$, and where $\sigma_{V'}$ is the volatility of the return of the reference firm value.

Once we have obtained the expression for w_t depending on the unobservable variables V'_t and $\sigma_{V'}$, we search for a relationthip between these variables and the price of the underlying stock and its return volatility. To do so, we use the fact that before debt maturity, shareholders and warrantholders own jointly a European call option on the value of the company, with strike equal to the face value of debt, and with exercise date T_D; that is, $NS_t + Mw_t = c(V'_t, F, T_D - t)$, where w_t is given by (55). Additionally, we use the expression $\sigma_S = \sigma_{V'}\Delta_S V'_t/S_t$.

Having related the unobservable and the observable variables, we can finally formulate the following proposition:

Proposition 3 *Let us consider a company with value denoted by V_t, and financed by N shares of stock, M European corporate call warrants with exercise date T, and a zero-coupon bond with face value F and maturity T_D, with $T_D < T$. For every warrant held, the warrant holder has the right to k shares in the company in exchange for payment of an amount X at time $t = T$. Let S_t be the stock price and let σ_S be the stock return volatility. Let V_t' be the value of a firm with the same investment policy as the warrant issuer but financed entirely by shares. For any time before the maturity of debt it is satisfied that the value of this firm and its return volatility are equal to the value of the levered firm and its volatility. Furthermore, if V_t' follows a geometric Brownian motion with standard deviation $\sigma_{V'}$ under a risk-neutral probability measure and in the absence of arbitrage opportunities, then the value at time t of a European call warrant on the company's shares will be given by the following algorithm:*

1. *Solve (numerically) the following system of non-linear equations for $(V_t'^*, \sigma_{V'}^*)$:*

$$\begin{cases} NS_t + M\frac{e^{-r(T_D-t)}}{\sqrt{2\pi(T_D-t)}}\int_{\bar{y}}^{\infty}\lambda c(V_{T_D}', F, T-T_D)e^{-\frac{y^2}{2}}dy = c(V_t', F, T_D - t) \\ \sigma_S = \sigma_{V'}\frac{\partial S_t}{\partial V_t'}\frac{V_t'}{S_t} \end{cases} \tag{56}$$

   where $c(x, K, T)$ denotes the value of a European call option on x, with strike K and time to maturity T, and with $\lambda = \frac{1}{N+kM}$, $y(V_{T_D}') = \frac{\ln\frac{V_{T_D}'}{V_t'} + (r - \frac{1}{2}\sigma_{V'}^2)(T_D - t)}{\sigma_{V'}\sqrt{T_D - t}}$ and $\bar{y} = \frac{\ln\frac{F}{V_t'} + (r - \frac{1}{2}\sigma_{V'}^2)(T_D - t)}{\sigma_{V'}\sqrt{T_D - t}}$.

2. *The warrant price at time t, with $t < T_D$, is obtained as:*

$$w_t(V_t'^*, \sigma_{V'}^*) = \frac{e^{-r(T_D-t)}}{\sqrt{2\pi(T_D-t)}}\Big(\int_{-\infty}^{\bar{y}}\lambda c(V_{T_D}', F, T-T_D)e^{-\frac{y^2}{2}}dy \tag{57}$$

4 Numerical examples

In this section we provide some applications of the warrant-pricing framework proposed in this paper to study its implementation. Specifically, we show various numerical applications comparing the results given by other warrant-pricing models.

First of all, in Tables 1 and 2 we study the application of the Ukhov (2004) algorithm by expanding Table 1 presented in his paper. In his table, Ukhov compares the prices given by three methods: the Black-Scholes-Merton formula, the Ingersoll (1987) pricing model and his own model. He investigates whether they are close for hypothetical warrants with different levels of dilution, different underlying stock prices and stock return variance. Parameters common for all calculations are $k = 1$, $X = 100$, $T - t = 3$, $r = 0.0488$ and $N = 100$. In our tables we add the warrant value given by the Ingersoll (1987) warrant-pricing formula, taking as values of V_t and σ_V the values V_t^* and σ_V^* that satisfy the system of equations (9), that is, the equilibrium values obtained for the reference firm and its volatility when using the observed value of the underlying stock and its volatility. We can check that the results for the warrant price given by this procedure are the same as we obtain when using the Ukhov (2004) formula taking as inputs the stock price, S_t, and its volatility, σ_S. Furthermore, we see that, as dilution increases, the warrant price decreases in all the cases except the Black and Scholes (1973) model, which ignores the dilution effect. We should also stress that the variation between the share volatility, σ_S, and the volatility obtained for firm value, σ_V^*, increases with increasing dilution.

In Tables 3 and 4 we compare the valuation of warrants using three models: the Black-Scholes-Merton formula, the Crouhy and Galai (1994) pricing model, and our own model when $T < T_D$. Parameters common for all calculations are now $k = 1$, $X = 100$, $T = 1$, $r = 0.0488$, $N = 100$, $F = 1000$ and $T_D = 3$. The second column gives the warrant prices given by the Black-Scholes-Merton formula. The third column shows the results given by the Crouhy and Galai model taking as initial values of V_t' and $\sigma_{V'}$ the values of NS_t and σ_S. From these values we find the reference asset value above which the warrants are exercised, $\bar{V}_T'$, which is the value of V_T' that satisfies $\frac{c(V_T'+100M,1000,2)}{N+M} = 100$, where $c(\cdot)$ is given by the Black and Scholes (1973) option pricing formula. Using the value of $\bar{V}_T'$ thus obtained, we simulate by Monte Carlo the value of V_t' from $t = 0$ to $t = T$. In each run, the firm value is determined as a function of whether the value of V_T' given by the simulation is below or above $\bar{V}_T'$, for which we use the expression of S_t given by (18). If the warrants are not exercised, the debt value at $t = T$ is $D_T^{NW} = V_T' - NS_T^{NW}$ and the warrant value is $w_T = 0$, whereas, if the warrants are exercised, we calculate the debt value as $D_T^W = V_T' + MX - (N + kM)S_T^W$ and the warrant value as $w_T = kS_T^W - X$.

Finally, after running 1.000.000 simulations, we obtain the values of S_t, D_t and w_t. With this valuation of the warrant at time $t = 0$, we have complemented the analysis performed by Crouhy and Galai, who implement their valuation model only for times close to the exercise date. Columns 4 - 6 show the results obtained with the algorithm presented in this paper for pricing levered warrants when $T < T_D$, which is implemented using the simulation described above and solving the system of non-linear equations given by (27) such that the value given by the simulation coincides with the known value of S_t and the expression of σ_S is satisfied. As before, we perform 1.000.000 simulations to obtain the warrant value, w_t. We show, in addition to the value of w_t obtained with our model, the values of V_t' and $\sigma_{V'}$ that solve the aforementioned system of equations, that is, the values of $V_t'^*$ and $\sigma_{V'}^*$. Finally, the seventh column shows the result obtained with the Crouhy and Galai (1994) model, using these values as values of V_t' and $\sigma_{V'}$. It can be seen that the value obtained for w_t is practically the same as that obtained with our algorithm. We should point out that, as in Tables 1 and 2, in both the Crouhy and Galai model and ours, the differences between the stock return volatility, σ_S, and the volatility obtained for the firm asset value, $\sigma_{V'}^*$, increase with increasing dilution. We should also mention, however, that in the case of low stock volatility and for warrants in the money or at the money, the value of each warrant decreases with increasing dilution with the Crouhy and Galai (1994) model but increases slightly with our valuation proposal. We must notice that while in the application of Crouhy and Galai with $V_t' = NS_t$ the value of the reference firm is invariant to changes in the degree of dilution, in the case of the implementation of our model, the value of the reference firm changes depending on the number of warrants and stocks outanding.

Finally, in Tables 5 and 6 we compare the valuation of warrants when $T = T_D$. Parameters are now $k = 1$, $X = 100$, $r = 0.0488$, $N = 100$, $F = 1000$ and $T = T_D = 3$[4]. The second column provides the warrant prices given by the Black-Scholes-Merton formula. The third column shows the results given by the Crouhy and Galai model taking as initial values of V_t and σ_V the values of NS_t and σ_S. The procedure followed to implement Crouhy and Galai (1994) in this case is the same as described for the case in which $T < T_D$.

[4]Since the implementation of the Crouhy and Galai (1994) model is not possible for $T = T_D$, we have taken T_D as 3.0000001.

Next, in columns 4 - 6 we show the results obtained with the algorithm presented in this paper for pricing levered warrants when $T = T_D$, given by expressions (42) - (48). In addition to the value obtained for w_t, we provide the values of V_t and σ_V that solve system (42), that is, V_t^* and σ_V^*. Finally, the seventh column shows the result obtained with the Crouhy and Galai (1994) model using these values as values of V_t and σ_V. It can be seen that the value obtained for w_t is practically the same as that obtained with our algorithm. We should remark that, as in Tables 1 - 4, in both the Crouhy and Galai model and ours, the differences between the stock return volatility, σ_S, and the volatility obtained for the firm asset value, σ_V^*, increase with dilution. Moreover, as in the case of $T < T_D$, we obtain that for low stock volatility and warrants in the money or at the money, the value of each warrant decreases with dilution with the Crouhy and Galai (1994) model but increases slightly with our valuation proposal. The reasoning for this fact is the same as before, that is, while in Crouhy and Galai (1994) the value of the reference firm is invariant to changes in the degree of dilution, in the valuation model we propose, the value of the reference firm changes depending on the number of warrants and stocks oustanding.

5 Conclusions

In this paper, we provide a valuation framework for pricing European call warrants on the issuer's own stock that takes debt into account. In contrast to other works which also price warrants with dilution issued by levered firms, ours uses only observable variables.

We consider three different cases depending on the exercise date: warrants expiring before debt maturity, warrants with the same maturity as debt and warrants with longer maturity than debt. In order to derive the valuation formula for each situation, and following Ukhov (2004), we first express the value of the warrant as a function of some unobservable variables. With the aim of obtaining such expression, we follow the Crouhy and Galai (1994) framework in the case of warrants with shorter and longer maturity than debt, and we draw on Ingersoll (1987) when the warrants have the same maturity as debt. Once obtained the expression for the warrant depending on unobservable variables, we relate these variables to the price of the underlying asset and its return volatility, whose values are observable.

Finally, to study the implementation of our valuation framework, we provide some numerical examples. Specifically, we provide various numerical applications comparing the results given by other warrant-pricing models, such as the Black-Scholes-Merton formula, the Crouhy and Galai (1994) model and the Ukhov (2004) algorithm. We study the prices given by the models for different levels of dilution, underlying stock prices and stock return variance.

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	BSM	**Ingersoll** (1)	**Ukhov's model**			**Ingersoll (2)**
	S_0, σ_S	$V = NS_0, \sigma_V = \sigma_S$	S_0, σ_S			$V = V_U^*, \sigma_V = \sigma_{V_U}^*$
S_0	w_{BSM}	w_{I1}	w_U	V_U^*	$\sigma_{V_U}^* (\%)$	w_{I2}
PANEL A. Low dilution, $N = 100$, $M = 10$						
75	8.8572	8.0520	8.8123	7588.12	25.82	8.8123
100	23.6712	21.5193	23.6834	10236.83	26.15	23.6834
110	31.1412	28.3102	31.1540	11311.54	26.20	31.1540
PANEL B. Medium dilution, $N = 100$, $M = 50$						
75	8.8572	5.9048	8.6528	7932.64	28.72	8.6528
100	23.6712	15.7808	23.6766	11183.83	30.14	23.6766
110	31.1412	20.7608	31.1363	12556.53	30.35	31.1363
PANEL C. High dilution, $N = 100$, $M = 100$						
75	8.8572	4.4286	8.4880	8348.79	31.73	8.4880
100	23.6712	11.8356	23.6064	12360.64	34.17	23.6064
110	31.1412	15.5706	31.0467	14104.66	34.47	31.0466

Low volatility, $\sigma_S = 25\%$

Table 1: Expansion of Table 1 of Ukhov (2004) for low volatility of stock returns. This table displays warrant prices computed by four methods: 1) w_{BSM} is the warrant price computed according to the Black-Scholes-Merton option formula; 2) w_{I1} corrects for dilution according to Ingersoll (1987), and $V = N \cdot S_0$, and $\sigma_V = \sigma_S$; 3) w_U is the warrant price obtained with Ukhov's model, and V_U^* and $\sigma_{V_U}^*$ are, respectively, the firm value and the standard deviation of the firm value process that satisfy system (9); and 4) w_{I2} is the warrant price computed with the Ingersoll's formula when firm value and its volatility are obtained with Ukhov's model. The remaining parameters are: $k = 1$, $X = 100$, $T = 3$ and $r = 0.0488$. Warrant prices are shown for three stock prices and three levels of dilution due to warrant exercise.

High volatility, $\sigma_S = 40\%$						
	BSM	**Ingersoll (1)**	**Ukhov's model**			**Ingersoll (2)**
	S_0, σ_S	$V = NS_0, \sigma_V = \sigma_S$	S_0, σ_S			$V = V_U^*, \sigma_V = \sigma_{V_U}^*$
S_0	w_{BSM}	w_{I1}	w_U	V_U^*	$\sigma_{V_U}^* (\%)$	w_{I2}
PANEL A. Low dilution, $N = 100$, $M = 10$						
75	16.6081	15.0983	16.5598	7665.59	41.30	16.5598
100	32.5992	29.6356	32.5671	10325.67	41.48	32.5672
110	39.9462	36.3148	39.9089	11399.09	41.51	39.9089
PANEL B. Medium dilution, $N = 100$, $M = 50$						
75	16.6081	11.0721	16.3742	8318.71	45.77	16.3741
100	32.5992	21.7328	32.3964	11619.82	46.51	32.3964
110	39.9462	26.6308	39.7088	12985.44	46.59	39.7088
PANEL C. High dilution, $N = 100$, $M = 100$						
75	16.6081	8.3040	16.1638	9116.38	50.26	16.1638
100	32.5992	16.2996	32.1503	13215.00	51.39	32.1503
110	39.9462	19.9731	39.4214	14942.14	51.46	39.4214

Table 2: Expansion of Table 1 of Ukhov (2004) for high volatility of stock returns. This table displays warrant prices computed by four methods: 1) w_{BSM} is the warrant price computed according to the Black-Scholes-Merton option formula; 2) w_{I1} corrects for dilution according to Ingersoll (1987), and $V = N \cdot S_0$, and $\sigma_V = \sigma_S$; 3) w_U is the warrant price obtained with Ukhov's model, and V_U^* and $\sigma_{V_U}^*$ are, respectively, the firm value and the standard deviation of the firm value process that satisfy system (9); and 4) w_{I2} is the warrant price computed with the Ingersoll's formula when firm value and its volatility are obtained with Ukhov's model. The remaining parameters are: $k = 1$, $X = 100$, $T = 3$ and $r = 0.0488$. Warrant prices are shown for three stock prices and three levels of dilution due to warrant exercise.

	Low volatility, $\sigma_S = 25\%$					
	BSM S_0, σ_S	**CG94** (1) $V' = NS_0, \sigma_{V'} = \sigma_S$	**AN model** S_0, σ_S			**CG94 (2)** $V' = V'^*_{AN}, \sigma_{V'} = \sigma^*_{V'_{AN}}$
S_0	w_{BSM}	w_{CG1}	w_{AN}	V'^*_{AN}	$\sigma^*_{V'_{AN}}$ (%)	w_{CG2}
PANEL A. Low dilution, $N = 100$, $M = 10$						
75	1.9052	0.7854	1.7877	8380.91	22.79	1.7864
100	12.2756	7.4141	12.2924	10984.49	24.15	12.2953
110	19.2280	12.5466	19.2930	12054.26	24.51	19.2908
PANEL B. Medium dilution, $N = 100$, $M = 50$						
75	1.9052	0.6374	1.6132	8444.51	24.06	1.6110
100	12.2756	5.5189	12.3502	11481.08	28.28	12.3441
110	19.2280	9.2909	19.3990	12833.56	29.32	19.3971
PANEL C. High dilution, $N = 100$, $M = 100$						
75	1.9052	0.4838	1.4382	8507.53	25.23	1.4387
100	12.2756	4.1468	12.3400	12095.90	32.70	12.3380
110	19.2280	6.9766	19.4316	13806.41	34.39	19.4311

Table 3: Pricing of levered warrants for low volatility of the stock return when $T < T_D$. This table displays warrant prices computed by four methods: 1) w_{BSM} is the warrant price computed according to the Black-Scholes-Merton option formula; 2) w_{CG1} corrects for dilution according to Crouhy-Galai (1994), and uses $V' = N \cdot S_0$, and $\sigma_{V'} = \sigma_S$; 3) w_{AN} is the warrant price obtained with the Abínzano-Navas approach when $T < T_D$; V'^*_{AN} and $\sigma^*_{V'_{AN}}$ are, respectively, the firm value and the standard deviation of the firm value process that satisfy system (27); and 4) w_{CG2} is the warrant price computed with Crouhy-Galai's formula when firm value and its volatility are obtained with the Abínzano-Navas model. The remaining parameters are: $k = 1$, $X = 100$, $T = 1$, $r = 0.0488$, $F = 1000$ and $T_D = 3$. Warrant prices are shown for three stock prices and three levels of dilution due to warrant exercise.

High volatility, $\sigma_S = 40\%$

	BSM	**CG94 (1)**	**AN model**			**CG94 (2)**
	S_0, σ_S	$V' = NS_0, \sigma_{V'} = \sigma_S$	S_0, σ_S			$V' = V'^{*}_{AN}, \sigma_{V'} = \sigma^{*}_{V'_{AN}}$
S_0	w_{BSM}	w_{CG1}	w_{AN}	V'^{*}_{AN}	$\sigma^{*}_{V'_{AN}}$ (%)	w_{CG2}
PANEL A. Low dilution, $N = 100$, $M = 10$						
75	5.6669	3.5519	5.4839	8420.77	36.80	5.5040
100	17.9693	12.7799	18.0276	11044.99	38.45	18.0558
110	24.6768	18.1191	24.8004	12112.95	38.88	24.8384
PANEL B. Medium dilution, $N = 100$, $M = 50$						
75	5.6669	2.6928	5.2266	8626.28	40.06	5.2361
100	17.9693	9.4893	17.9858	11760.20	44.20	17.9892
110	24.6768	13.4165	24.8051	13100.56	45.18	24.8075
PANEL C. High dilution, $N = 100$, $M = 100$						
75	5.6669	2.0278	5.0064	8862.83	43.44	4.9983
100	17.9693	7.1280	17.9171	12652.83	50.20	17.9291
110	24.6768	10.0745	24.7653	14335.71	51.63	24.7613

Table 4: Pricing of levered warrants for high volatility of the stock return when $T < T_D$. This table displays warrant prices computed by four methods: 1) w_{BSM} is the warrant price computed according to the Black-Scholes-Merton option formula; 2) w_{CG1} corrects for dilution according to Crouhy-Galai (1994), and uses $V' = N \cdot S_0$, and $\sigma_{V'} = \sigma_S$; 3) w_{AN} is the warrant price obtained with the Abínzano-Navas approach when $T < T_D$; V'^{*}_{AN} and $\sigma^{*}_{V'_{AN}}$ are, respectively, the firm value and the standard deviation of the firm value process that satisfy system (27); and 4) w_{CG2} is the warrant price computed with Crouhy-Galai's formula when firm value and its volatility are obtained with the Abínzano-Navas model. The remaining parameters are: $k = 1$, $X = 100$, $T = 1$, $r = 0.0488$, $F = 1000$ and $T_D = 3$. Warrant prices are shown for three stock prices and three levels of dilution due to warrant exercise.

	BSM	**CG94 (1)**	**AN model**			**CG94 (2)**
	S_0, σ_S	$V' = NS_0, \sigma_{V'} = \sigma_S$	S_0, σ_S			$V' = V_{AN}, \sigma_{V'} = \sigma^*_{V_{AN}}$
S_0	w_{BSM}	w_{CG1}	w_{AN}	V^*_{AN}	$\sigma^*_{V_{AN}}(\%)$	w_{CG2}
PANEL A. Low dilution, $N = 100$, $M = 10$						
75	8.8572	6.0650	8.7391	8451.20	23.16	8.7621
100	23.6712	17.6025	23.7982	11101.79	24.10	23.8415
110	31.1412	23.7167	31.3129	12176.94	24.33	31.3615
PANEL B. Medium dilution, $N = 100$, $M = 50$						
75	8.8572	4.4273	8.5739	8792.51	25.79	8.5739
100	23.6712	12.8813	23.8083	12054.23	27.88	23.8141
110	31.1412	17.3624	31.3195	13429.78	28.30	31.3291
PANEL C. High dilution, $N = 100$, $M = 100$						
75	8.8572	3.3186	8.4041	9204.22	28.54	8.4032
100	23.6712	9.6584	23.7540	13239.21	31.74	23.7576
110	31.1412	13.0190	31.2473	14988.54	32.28	31.2523

Low volatility, $\sigma_S = 25\%$

Table 5: Pricing of levered warrants for low volatility of the stock return when $T = T_D$. This table displays warrant prices computed by four methods: 1) w_{BSM} is the warrant price computed according to the Black-Scholes-Merton option formula; 2) w_{CG1} corrects for dilution according to Crouhy-Galai (1994), and uses $V' = N \cdot S_0$, and $\sigma_{V'} = \sigma_S$; 3) w_{AN} is the warrant price obtained with the Abínzano-Navas approach when $T = T_D$; V^*_{AN} and $\sigma^*_{V_{AN}}$ are, respectively, the firm value and the standard deviation of the firm value process that satisfy system (42); and 4) w_{CG2} is the warrant price computed with Crouhy-Galai's formula when firm value and its volatility are obtained with the Abínzano-Navas model. The remaining parameters are: $k = 1$, $X = 100$, $r = 0.0488$, $F = 1000$ and $T = T_D = 3$ (3.0000001 in the Crouhy and Galai (1994) model since it does not allow T to be equal to T_D). Warrant prices are shown for three stock prices and three levels of dilution due to warrant exercise.

	High volatility, $\sigma_S = 40\%$					
	BSM	**CG94 (1)**	**AN model**			**CG94 (2)**
	S_0, σ_S	$V' = NS_0, \sigma_{V'} = \sigma_S$	S_0, σ_S			$V' = V_{AN}, \sigma_{V'} = \sigma^*_{V_{AN}}$
S_0	w_{BSM}	w_{CG1}	w_{AN}	V^*_{AN}	$\sigma^*_{V_{AN}}(\%)$	w_{CG2}
PANEL A. Low dilution, $N = 100$, $M = 10$						
75	16.6081	13.1133	16.4939	8528.68	37.07	16.6045
100	32.5992	20.5033	32.8007	11191.79	38.24	32.9597
110	39.9462	32.7731	40.2254	12266.05	38.55	40.4064
PANEL B. Medium dilution, $N = 100$, $M = 50$						
75	16.6081	9.5438	16.3116	9179.16	41.22	16.3280
100	32.5992	19.3389	32.6573	12496.58	43.07	32.6877
110	39.9462	23.9270	40.0572	13866.60	43.47	40.0925
PANEL C. High dilution, $N = 100$, $M = 100$						
75	16.6081	7.1511	16.1071	9973.96	45.43	16.1136
100	32.5992	14.4951	32.4353	14107.09	47.81	32.4525
110	39.9462	17.9353	39.7961	15843.24	48.24	39.8155

Table 6: Pricing of levered warrants for high volatility of the stock return when $T = T_D$. This table displays warrant prices computed by four methods: 1) w_{BSM} is the warrant price computed according to the Black-Scholes-Merton option formula; 2) w_{CG1} corrects for dilution according to Crouhy-Galai (1994), and uses $V' = N \cdot S_0$, and $\sigma_{V'} = \sigma_S$; 3) w_{AN} is the warrant price obtained with the Abínzano-Navas approach when $T = T_D$; V^*_{AN} and $\sigma^*_{V_{AN}}$ are, respectively, the firm value and the standard deviation of the firm value process that satisfy system (42); and 4) w_{CG2} is the warrant price computed with Crouhy-Galai's formula when firm value and its volatility are obtained with the Abínzano-Navas model. The remaining parameters are: $k = 1$, $X = 100$, $r = 0.0488$, $F = 1000$ and $T = T_D = 3$ (3.0000001 in the Crouhy and Galai (1994) model since it does not allow T to be equal to T_D). Warrant prices are shown for three stock prices and three levels of dilution due to warrant exercise.