Single and Double Black-Cox for pricing risky debt and equity with reorganization*

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Abstract

In this paper we add a more realistic first-passage-time approach for pricing debt and equity when the firm has the possibility of carrying on a reorganization as an alternative to liquidation. In contrast to other first passage models that account for reorganization, our approach allows the firm to restructure its debt, by changing its maturity and/or its face value. Furthermore, our approach considers two types of settings. The first one is a first-passage approach for reorganization together with a Merton approach for default, while the second setting uses first-passage models for both reorganization and default. Finally, a comparison among the models proposed with Merton (1974) and Black and Cox (1976) approaches is provided.

Journal of Economic Literature classification: G12, G32, G33.

Keywords: First passage time model, reorganization, liquidation.

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*We would like to thank Manuel Moreno and seminar participants at the XV Foro de Finanzas and the XXXII Simposio de Análisis Económico for their valuable suggestions and comments. The first author gratefully acknowledges financial support from the Spanish Ministry of Education and Science (MEC, SEJ2006-14809-C03-01) and from a grant from Fundación ICO, and thanks the University of Toronto, where part of this work was done, for its hospitality. The usual caveat applies.

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1 Introduction

Corporate bankruptcy is very common. Indeed, hundreds of thousands of firms around the world declare bankruptcy each year. It is said that a company experiences financial distress when it faces insufficient liquidity to meet its financial liabilities.

When a corporation becomes insolvent and bankruptcy proceedings are commenced, the corporation may take two routes: liquidation or reorganization. In liquidation, the assets of the corporation are sold, either piecemeal or as a going concern. The proceeds from this sale are then divided among those who have rights against the corporation, with the division made according to the ranking of these rights. Alternatively, reorganization consists of a series of agreements between the company and its creditors which allow for the company to repay its debts and alter its structure to prevent the same problems from arising again. Thus, it can be seen as an attempt to extend the life of the company facing bankruptcy through special arrangements and restructuring.

In order to restructure their debt, firms have traditionally used either an out of court workout (public of private) or a formal Chapter 11 bankruptcy[1]. Chapter 11 provides several benefits for distressed firms, allowing them to continue operations without the "harassment" of creditors (see Gertner and Scharfstein, 1991). The main purpose of Ch. 11 is to preserve the company as an operating concern while a plan for reorganization is worked out among creditors. However, the primary disadvantage of a Chapter 11 reorganization is its relative cost. Gilson, John and Lang (1990) and Jensen (1989) suggest that due to high restructuring costs of Chapter 11, firms have and incentive to reorganize voluntarily out of court via a workout. However, workouts are thwarted by holdout problems, as shown by Chatterjee, Dhillon and Ramírez (1995). An alternative to Chapter 11 and workouts is the prepackaged bankruptcy plan. A prepackaged bankruptcy (a "prepack") is a reorganization of firm's debt contracts that has been negotiated or accepted by creditors prior to beginning of a bankruptcy proceeding. The idea behind a pre-packaged bankruptcy plan is to shorten and simplify the bankruptcy process in order to save the company money in legal fees, and hopefully generate more for the creditors. Specifically, prepacks eliminate the holdout problem associated with workouts; at the same time, they avoid a prolonged stay in Chapter 11.

[1]Hereafter we are using Chapter 11 of the U.S. Bankruptcy Code to refer to a formal reorganization procedure, although we might use other formal procedures, such as the one introduced by the Canadian Bankruptcy and Insolvecy Act (see Fisher and Martel 2003).

Chatterjee et al. (1996) find that the restructuring decision depends on the degree of the firm's leverage, the severity of its liquidity crisis, the extent of creditor's coordination and the magnitude of its economic distress. Concretely, they obtain that economically distressed firms file for Chapter 11, while economically viable firms prefer workouts. Further, prepackaged bankruptcies are used by firms that are economically viable but face immediate liquidity problems.

Independently of the procedure used to restructure debt, the rationale commonly offered is that a reorganization may enable the participants to capture a greater value than they can obtain in a liquidation. According to Bebchuk (1988), reorganization is thought to be specially valuable when the company's assets are worth much more as a going concern than if sold piecemeal, and when there are few or even no outside buyers with both accurate information about the company and sufficient resources to acquire it. In such situations, the participants will have more value to split if they retain the enterprise and divide it among themselves. On the other hand, Brown (1989) argues that when a firm is in financial distress, reorganization is often desirable because it avoids losses form liquidation or suboptimal incentive alignment of the existing contracts.

The modeling of default and reorganization is instrumental in determining the values of corporate securities and the firm's financing decisions since it affects firm value and its sharing among claimholders. In the literature we can find several works that consider the possibility of a reorganization in the pricing of corporate securities. Hence, Franks and Torous (1989) provide a framework where the value of equity at maturity can be seen as an American call option on the firm's assets value. Longstaff (1990) points out that many types of corporate reorganizations -such as Chapter 11 bankruptcy- can be viewed as the exercise of an extension privilege and suggests the application of the extendible option analysis to the pricing of the capital structure of a firm. Furthermore, Longstaff (1990) applies the flexible writer-extendible options to the pricing of the equity of a risky levered firm where bondholders have an incentive to extend the maturity date of the debt, whereas Abínzano and Navas (2007) apply the holder-extendible options defined by Longstaff (1990) to the pricing of the equity position of firms where shareholders have the possibility of filing a plan of debt restructuring. Recently, Dumitrescu (2007) proposes a contingent valuation model for a firm financed by two bonds with different maturities and different seniorage that allows to analyze the implications of debt renegotiation on the prices of bonds.

Notwithstanding, all these papers only permit reorganization (or default) at maturity. That means that the firm must wait until maturity to default or carry on a reorganization. In other words, the value of the firm is allowed to dwindle to nearly nothing without triggering a default. Black and Cox (1976) propose a more realistic model by allowing premature default. In their model, the default occurs as soon as the value of the firm's assets reaches a lower threshold. Thus, the default time is defined by the first passage time of the firm value to some barrier.

Nevertheless, the Black and Cox model does not account for the possibility of a reorganization as an alternative to liquidation. Recently, two first-passage-time models have been proposed to consider the possibility of carrying a reorganization. Fan and Sundaresan (2000) provide a framework of reorganization where the claimants negotiate at a certain (endogenous) threshold to accept a reduced level of debt service until the asset value goes back above the threshold again. Afterwards, and following Fan and Sundaresan (2000), Francois and Morellec (2004) present a contingent claims model where the decision for liquidating or reorganizing the firm depends only on the length of the period of time after crossing the threshold.

As it can be noticed, reorganization in Fan and Sundaresan (2000) and Francois and Morellec (2004) consists of a reduction on the payments serviced by equity holders during the period of reorganization. In contrast, a more realistic setting would be that as soon as the firm assets value falls to some prescribed lower threshold, the firm could restructure its debt, for example forcing Chapter 11, by changing its maturity and/or its face value. For this reason, in this paper we propose a model for pricing debt and equity when the firm has the possibility of reorganization inspired on Black-Cox (1976), that is, restructuring or defaulting is allowed to occur before maturity as soon as some pre-agreed thresholds are exceeded, and where reorganization consists of an extension of the debt maturity and/or a modification of the amount due.

The rest of the paper is organized as follows. Section 2 provides the main mathematical results needed to develop the pricing model. In Section 3 a single Black-Cox (1976) approach for reorganization is proposed. In the same section, an extension to allow for first-passage model for both reorganization and default is added. A comparison of these approaches is provided in Section 4. Finally, Section 5 concludes.

2 Mathematical setting

As we have seen before, the purpose of this paper is to develop a model for pricing the claims of a company with the possibility of carrying on a reorganization as soon as its assets value falls to some lower threshold. Next, we present some mathematical setting we would need to develop the model.

The mathematical results needed are related to the joint distribution of the endpoint and the minimum of a Brownian motion with drift. The importance of this joint lies in the equivalence between the time of default of a company and the minimum of the value of its assets. Therefore, several distributions related to the minimum of a lognormal process with constant drift and volatility are provided in the next two propositions.

Consider the following notation:

1. V_t stands for the value of the assets of a company, while $X_t = \ln V_t$.
2. V_R denotes the reorganization threshold, with $X_R = \ln\ V_R$.
3. V_L denotes the liquidation threshold, with $X_L = \ln\ V_L$, $V_R > V_L$.
4. K_1 stands for the face value of debt, with $V_R \leq K_1$.
5. K_2 is the "re-assessed" debt face value, after debt has been reorganized, with $V_L \leq K_2$. Moreover, we assume that K_2 can be larger or lesser than K_1.

In deriving the next mathematical results, we assume the following conditions:

1. There is no recovery in the presence of default.
2. The interest rate, r, is known and constant through time.
3. The standard deviation of the return on the assets value, σ, is constant.

Thus, we reach the next proposition.

Proposition 1. *Let X_t be a stochastic process defined as follows:*

$$X_t = \mu t + \sigma W_t,\ t > c,\ X_c = X_0 \tag{1}$$

$$Mn_t = \min_{c \leq s \leq t} X_s \tag{2}$$

$$\tau_1 = \inf\{t > c : X_t = X_R\} \tag{3}$$

$$X_R < X_0 \tag{4}$$

Where W_t stands for a standard Brownian motion and c is the initial date. Based on He et al. (1998), we obtain the following results:

1. *The cumulative of τ_1 is:*

$$P(\tau_1 > t) = P(Mn_t > X_R) = P(X_t > X_R, Mn_t > X_R) =$$

$$= P(X_t > X_R) - P(X_t > X_R, Mn_t < X_R)$$

$$= \Phi\left(\frac{X_0 - X_R + \mu(t-c)}{\sigma\sqrt{(t-c)}}\right) - \exp\left(\frac{2\mu(X_R - X_0)}{\sigma^2}\right)\Phi\left(\frac{X_R - X_0 + \mu(t-c)}{\sigma\sqrt{(t-c)}}\right) \tag{5}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Normal.

2. *Differentiating with respect to t we obtain the density of τ_1:*

$$f_{\tau_1}(t) = \phi\left(\frac{X_0 - X_R + \mu(t-c)}{\sigma\sqrt{(t-c)}}\right)\left(\frac{X_0 - X_R}{2\sigma(t-c)^{3/2}} - \frac{\mu}{2\sigma\sqrt{(t-c)}}\right)$$

$$-\exp\left(\frac{2\mu(X_R - X_0)}{\sigma^2}\right)\phi\left(\frac{X_R - X_0 + \mu(t-c)}{\sigma\sqrt{(t-c)}}\right)\left(\frac{X_R - X_0}{2\sigma(t-c)^{3/2}} - \frac{\mu}{2\sigma\sqrt{(t-c)}}\right) \tag{6}$$

where $\phi(\cdot)$ is the density function of the standard Normal distribution.

3. *The joint density/distribution function of (X_t, τ_1), $p_{X,\tau_1}(x,t)dx$, can be obtained using Fokker-Planck equations ($x \geq X_R$):*

$$P(X_t \in dx, \tau_1 > t) = p_{X,\tau_1}(x,t)dx =$$

$$= \frac{\phi\left(\frac{x-\mu(t-c)}{\sigma\sqrt{(t-c)}}\right)}{\sigma\sqrt{(t-c)}}\left[1 - \exp\left(\frac{-\left(4(X_R - X_0)^2 - 4(X_R - X_0)x\right)}{2\sigma^2(t-c)}\right)\right]dx \tag{7}$$

In addition, we may make some some observations about the previous results:

Remark 1. *1. If $\mu > 0$, then $P(\tau_1 > \infty) = 1 - \exp\left(\frac{2\mu(X_R - X_0)}{\sigma^2}\right)$, that is, there is a positive probability of never crossing the threshold X_R.*

2. *If* $\mu < 0$*, then* $P(X_R \leq X_t < \infty, \tau_1 < \infty) \approx 0$*, what means there is a small probability of both events, crossing a threshold and the endpoint being greater than* X_R*, happening simultaneously. The reason for this is that* $P(X_R \leq X_t < \infty) \approx 0$*.*

3. *The assumption of constant volatility is required for the proposition to apply. On the other hand, the assumptions of zero recovery and constant interest rate could be relaxed, what would require to use some challenging, from the computational side, joint distributions, which could be found in He et al. (1998).*

As it can be inferred from Proposition 1, the variable τ_1 accounts for the time where the reorganization threshold is crossed. Another important variable is the time where the liquidation threshold is crossed, τ_2. Let us define:

$$\tau_2 = \inf\{t > \tau_1 : X_t = X_L\},\ X_L < X_R \tag{8}$$

Consequently, we can enunciate the following result:

Proposition 2. *The joint cumulative distribution for* (τ_1, τ_2) *is:*

$$P(\tau_1 < t_1, \tau_2 < t_2) = \int_0^{t_1} P(\tau_2 < t_2 | \tau_1 = s_1) f_{\tau_1}(s_1) ds_1 \tag{9}$$

$$P(\tau_2 < t_2 | \tau_1 = s_1) = \Phi\left(\frac{X_R - X_L + \mu(t_2 - s_1)}{\sigma\sqrt{t_2 - s_1}}\right) - \exp\left(\frac{2\mu(X_L - X_R)}{\sigma^2}\right)\Phi\left(\frac{X_L - X_R + \mu(t_2 - s_1)}{\sigma\sqrt{t_2 - s_1}}\right) \tag{10}$$

where $f_{\tau_1}(s_1)$ *is the density function found in Proposition 1 while* $P(\tau_2 < t | \tau_1 = s_1)$ *is the density function of a Brownian motion starting at* $X_{s_1} = X_R$ *with threshold* X_L*.*

Finally, we should remark that differentiating equation (9) with respect to t_1 and t_2 would lead to the joint density of (τ_1, τ_2), $f_{\tau_1,\tau_2}(t_1, t_2)$.

3 The valuation framework

In this section, we develop two extensions of the Black and Cox (1976) model with the aim of accounting for the possibility of a reorganization. The first extension is a first-passage-time approach for reorganization together with a Merton approach for default, while the second extension uses first-passage-time models for both reorganization and default.

Let us consider a firm financed with shares of stock and one zero-coupon bond with face value K_1 and maturity T_1. Let us suppose there is no possibility of reorganization. In this context, Merton (1974) proposes to consider the value of the equity as a call option on the value of the assets of the firm with strike price equal to the face value of debt. In this approach, default of the company occurs only at maturity T_1. As a result, the payoff expected by shareholders at T_1 shall be:

$$\left(V_{T_1} - K_1\right) 1_{\{V_{T_1} > K_1\}} \tag{11}$$

while the payoff at maturity from the view of debtholders shall be:

$$K_1 1_{\{V_{T_1} > K_1\}} + V_{T_1} 1_{\{V_{T_1} \leq K_1\}} \tag{12}$$

These payoffs are random variables. Thus, if the value of the company is larger than its debt, debtholders get K_1 while shareholders get $V_{T_1} - K_1$, and in case of default, debtholders receive what is left of the company, V_{T_1}, while shareholders get 0.

In contrast to Merton (1974), Black and Cox (1976) allow default to occur at anytime previous to maturity. If we consider a firm financed with equity and one zero-coupon bond with face value K_1 and maturity T_1, the payoff expected by shareholders at $\min(T_1, \tau_1)$ shall be:

$$(V_{T_1} - K_1) 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} > K_1\}} \tag{13}$$

while from the view of debtholders, the payoff at $\min(T_1, \tau_1)$ shall be:

$$K_1 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} > K_1\}} + V_{T_1} 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} \leq K_1\}} + V_L 1_{\{\tau_1 \leq T_1\}} \tag{14}$$

with $V_L \leq K_1$. In other words, as soon as the value of the company is lower than its debt, debtholders get K_1 while shareholders get 0. On the contrary, if no default occurs, shareholders get $V_{T_1} - K_1$ while debtholders keep getting the value of the debt, K_1.

Let us assume now that the firm has the possibility of carrying a reorganization. Concretely, we define a reorganization as an extension of the debt maturity and/or a modification of the amount due. In the case of a formal procedure- such as a Chapter 11 filing- the debtor is entitled to file a reorganization plan during a specified period of time. Creditors will accept this reorganization plan only if the value of debt after reorganization is higher than the payments they would receive in case of liquidation. When liquidation occurs, the control of the firm is transferred from the equity to the debtholders. It is commonly accepted that this transfer is costly. Indeed, Warner (1977) distinguishes between two types of costs: direct and indirect. Direct costs include lawyers and accountants fees, while indirect costs include lost sales and lost profits. In what follows we will assume that costs of liquidation are high enough to debtholders to accept the reorganization plan. This assumption is consistent with Branch (2002), who points out that a significant part of the value of the bankrupt firm is consumed in dealing with its distress. Furthermore, Altman (1984) finds that the total (direct and indirect) cost of liquidation amount to about 15 % of predistress firm value.

3.1 Single Black-Cox risky debt with reorganization

Consider the firm described above. Let us make the following assumptions:

1. A reorganization is allowed as soon as the asset's value crosses a threshold, V_R, with $\tau_1 \leq T_1$. The maturity time as well as the face value of debt are changed at this stage to T_2 and K_2, respectively.

2. Default may occur only at T_2 if and only if $V_{T_2} < K_2$.

From Section 2 it is also assumed that $V_R \leq K_1$, which means that reorganization makes sense if and only if the assets of the company are not enough to cover the debt. We must also remark that the new face value of debt, K_2, could be less than the initial face value, K_1. This is consistent with Haugen and Senbet (1978, 1988), who argue that debtholders could offer to reduce the debt claim so as to avoid bankruptcy and its associated costs.

We must remark that, in contrast to Fan and Sundaresan (2000) and Francois and Morellec (2004), in our setting the distress thresholds, V_R and V_L, are inputs to the problem. As noted by Leland (1994) and Ericson and Reneby (1998), there are several ways to determine and justify a distress threshold. An economic approach views the distress threshold as the level of assets value necessary for the firm to retain sufficient credibility to continue operations. A contractual interpretation for the existence of the distress threshold is based on positive net worth covenants that enable bondholders to force reorganization or liquidation in the event the value of the firm falls below a pre-determined threshold. Thus, we can see V_R and V_L as wake-up calls to the company that the value of its assets are too low to keep operating in normal conditions. An example could be to take V_R and V_L as K_1 and K_2, respectively.

3.1.1 Shareholders viewpoint

In the Single Black-Cox framework, the payoff condition satisfied at $\min(T_1, \tau_1)$ from the point of view of shareholders (SH) is:

$$(V_{T_1} - K_1)1_{\{\tau_1 > T_1\}}1_{\{V_{T_1} > K_1\}} + c(V_R, K_2, T_2 - \tau_1)1_{\{\tau_1 \leq T_1\}} \tag{15}$$

where $c(V_R, K_2, T_2 - \tau_1)$ denotes the price of a European call option.

Proposition 3. *Under previous assumptions, the price of a single Black-Cox equity with reorganization is:*

$$\begin{aligned}
&SBC_{SH}(V_0, V_R, K_1, K_2, T_1, T_2) = \\
&= e^{-rT_1}E_0\left[(\exp(X_{T_1}) - K_1)1_{\{\tau_1 > T_1\}}1_{\{X_{T_1} > \ln K_1\}}\right] \\
&+E_0\left[e^{-r\tau_1}c(V_R, K_2, T_2 - \tau_1)1_{\{\tau_1 < T_1\}}\right] = \\
&= e^{-rT_1}\int_{\ln K_1}^{\infty}(\exp(x) - K_1)p_{X,\tau_1}(x, T_1)dx + \int_0^{T_1}e^{-rt}c(V_R, K_2, T_2 - t)f_{\tau_1}(t)dt
\end{aligned} \tag{16}$$

where:

$$c(V_R, K_2, T_2 - t) = V_R\Phi(a) - K_2e^{-r(T_2-t)}\Phi(b) \tag{17}$$

with:

$$a = \frac{\ln\frac{V_R}{K_2} + \left(r + \frac{1}{2}\sigma^2\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}} \tag{18}$$

$$b = a - \sigma\sqrt{T_2 - t} \tag{19}$$

and with $f_{\tau_1}(t)$ and $p_{X,\tau_1}(x, T_1)$ defined by (6) and (7).

Proof. This follows by substituting properly from Proposition 1. □

Remark 2. *The "price of reorganizing" for shareholders within Black-Cox framework would be the difference between the value of SBC and BC:*

$$E_0\left[e^{-r\tau_1}c(V_R, K_2, T_2 - \tau_1)1_{\{\tau_1 \leq T_1\}}\right] \tag{20}$$

3.1.2 Debtholders viewpoint

On the other hand, the payoff condition satisfied at $\min(T_1, \tau_1)$ from the point of view of debtholders (DB) is:

$$K_1 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} > K_1\}} + V_{T_1} 1_{\{\tau_1 > T_1\}} 1_{\{V_{T_1} \leq K_1\}} +$$
$$\left(K_2 e^{-r(T_2-\tau_1)} - p(V_R, K_2, T_2 - \tau_1)\right) 1_{\{\tau_1 \leq T_1\}} \tag{21}$$

where $p(V_R, K_2, T_2 - \tau_1)$ denotes the price of a European put option.

Proposition 4. *Under previous assumptions the price of a single Black-Cox risky debt with reorganization is:*

$$SBC_{DB}(V_0, V_R, K_1, K_2, T_1, T_2) =$$
$$= e^{-rT_1} K_1 E_0 \left[1_{\{\tau_1 > T_1\}} 1_{\{X_{T_1} > \ln K_1\}}\right] + e^{-rT_1} E_0 \left[\exp(X_{T_1}) 1_{\{\tau_1 > T_1\}} 1_{\{X_{T_1} \leq \ln K_1\}}\right]$$
$$+ K_2 E_0 \left[e^{-rT_2} 1_{\{\tau_1 < T_1\}}\right] - E_0 \left[e^{-r\tau_1} p(V_R, K_2, T_2 - \tau_1) 1_{\{\tau_1 < T_1\}}\right]$$
$$= e^{-rT_1} K_1 \int_{\ln K_1}^{\infty} p_{X,\tau_1}(x, T_1) dx + e^{-rT_1} \int_{-\infty}^{\ln K_1} \exp(x) p_{X,\tau_1}(x, T_1) dx$$
$$+ K_2 e^{-rT_2} \int_0^{T_1} f_{\tau_1}(t) dt - \int_0^{T_1} e^{-rt} p(V_R, K_2, T_2 - t) f_{\tau_1}(t) dt \tag{22}$$

where:

$$p(V_R, K_2, T_2 - t) = K_2 e^{-r(T_2-t)} \Phi(-b) - V_R \Phi(-a) \tag{23}$$

with:

$$a = \frac{\ln\left(\frac{V_R}{K_2}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}} \tag{24}$$

$$b = a - \sigma\sqrt{T_2 - t} \tag{25}$$

and with $f_{\tau_1}(t)$ and $p_{X,\tau_1}(x, T_1)$ defined by (6) and (7).

Proof. This follows by substituting properly from Proposition 1. □

Remark 3. *The "price of reorganizing" for debtholders within Black-Cox framework would be the difference between the value of SBC and BC:*

$$E_0 \left[e^{-r\tau_1} \left(K_2 e^{-r(T_2-\tau_1)} - p(V_R, K_2, T_2 - \tau_1) - V_R\right) 1_{\{\tau_1 \leq T_1\}}\right] \tag{26}$$

3.2 Double Black-Cox risky debt with reorganization

This approach consists of a generalization of the Single Black-Cox in the sense that after reorganization the default may occur at anytime before the newly set maturity time. As before, there would be two main assumptions in this case:

1. A reorganization is allowed as soon as the value of the assets crosses a threshold, V_R, with $\tau_1 \leq T_1$. Both the maturity time and face value of debt are changed at this stage to T_2 and K_2, respectively.

2. Default may occur either before maturity T_2 as soon as $V_t < V_L$, with $V_L < V_R$ and $t > \tau_1$, or it can occur at maturity if $V_{T_2} < K_2$.

3.2.1 Shareholders viewpoint

Now, in the Double Black-Cox framework the payoff condition satisfied at $\min(T_1, \tau_1)$ from the point of view of shareholders is:

$$(V_{T_1} - K_1)1_{\{\tau_1 > T_1\}}1_{\{V_{T_1} > K_1\}} + cFPM(V_R, K_2, V_L, T_2 - \tau_1)1_{\{\tau_1 < T_1\}} \tag{27}$$

where $cFPM(V_R, K_2, V_L, T)$ is a first passage model European option and $V_R = V(0)$, that is:

$$\begin{aligned}
&cFPM(V_R, K_2, V_L, T) = \\
&= e^{-rT} E\left[(\exp(X_T) - K_2)1_{\{\tau_2 > T\}}1_{\{X_T > \ln K_2\}}\right] \\
&= e^{-rT} \int_{\ln K_2}^{\infty} (\exp(x) - K_2) p_{X,\tau_2}(x, T) dx
\end{aligned} \tag{28}$$

Proposition 5. *Under previous assumptions, the price of a double Black-Cox equity with reorganization is:*

$$\begin{aligned}
&DBC_{SH}(V_0, V_R, V_L, K_1, K_2, T_1, T_2) \\
&= e^{-rT_1} E_0\left[(\exp(X_{T_1}) - K_1)1_{\{\tau_1 > T_1\}}1_{\{X_{T_1} > \ln K_1\}}\right] \\
&+ E_0\left[e^{-r\tau_1} cFPM(V_R, K_2, V_L, T_2 - \tau_1)1_{\{\tau_1 < T_1\}}\right] \\
&= e^{-rT_1} \int_{\ln K_1}^{\infty} (\exp(x) - K_1) p_{X,\tau_1}(x, T_1) dx \\
&+ e^{-rT_2} \int_0^{T_1} \int_{\ln K_2}^{\infty} (\exp(x) - K_2) p_{X,\tau_2}(x, T_2 - t_1) f_{\tau_1}(t_1) dx dt_1
\end{aligned} \tag{29}$$

Proof. This follows by substituting properly from Propositions 1 and 2. □

Remark 4. *The "price of reorganizing" for shareholders within Black-Cox framework would be the difference between the value of DBC and BC:*

$$E_0\left[e^{-r\tau_1}cFPM(V_R,K_2,V_L,T_2-\tau_1)1_{\{\tau_1<T_1\}}\right] \tag{30}$$

3.2.2 Debtholders viewpoint

On the other hand, the payoff condition at $\min(T_1,\tau_1)$ from the point of view of debtholders is:

$$\begin{aligned} &K_1 1_{\{\tau_1>T_1\}}1_{\{V_{T_1}>K_1\}}+V_{T_1}1_{\{\tau_1>T_1\}}1_{\{V_{T_1}\le K_1\}}+ \\ &+E_{\tau_1}\left[\begin{array}{c} e^{-r(T_2-\tau_1)}K_2 1_{\{\tau_2>T_2\}}1_{\{V_{T_2}>K_2\}} \\ +e^{-r(T_2-\tau_1)}V_{T_2}1_{\{\tau_2>T_2\}}1_{\{V_{T_2}\le K_2\}}+e^{-r\tau_2}V_L 1_{\{\tau_2\le T_2\}} \end{array}\right]\cdot 1_{\{\tau_1\le T_1\}} \end{aligned} \tag{31}$$

Proposition 6. *The price of a double Black-Cox risky debt with reorganization is:*

$$\begin{aligned} &DBC_{DH}(V_0,V_R,V_L,K_1,K_2,T_1,T_2)= \\ &=e^{-rT_1}K_1E_0\left[1_{\{\tau_1>T_1\}}1_{\{X_{T_1}>\ln K_1\}}\right]+e^{-rT_1}E_0\left[\exp(X_{T_1})1_{\{\tau_1>T_1\}}1_{\{X_{T_1}<\ln K_1\}}\right] \\ &+K_2e^{-rT_2}E_0\left[1_{\{\tau_1<T_1\}}1_{\{\tau_2>T_2\}}1_{\{X_{T_2}>\ln K_2\}}\right] \\ &+e^{-rT_2}E_0\left[\exp(X_{T_2})1_{\{\tau_1<T_1\}}1_{\{\tau_2>T_2\}}1_{\{X_{T_2}<\ln K_2\}}\right]+V_LE_0\left[e^{-r\tau_2}1_{\{\tau_1<T_1\}}1_{\{\tau_2<T_2\}}\right] \\ &=e^{-rT_1}K_1\int_{\ln K_1}^{\infty}p_{X,\tau_1}(x,T_1)dx+e^{-rT_1}\int_{-\infty}^{\ln K_1}\exp(x)p_{X,\tau_1}(x,T_1)dx \\ &+e^{-rT_2}K_2\int_0^{T_1}\left(\int_{\ln K_2}^{\infty}p_{X,\tau_2}(x,T_2-t_1)dx\right)f_{\tau_1}(t_1)dt_1 \\ &+e^{-rT_2}\int_0^{T_1}\left(\int_{-\infty}^{\ln K_2}\exp(x)p_{X,\tau_2}(x,T_2-t_1)dx\right)f_{\tau_1}(t_1)dt_1 \\ &+V_L\int_0^{T_1}\left(\int_0^{T_2-t_1}e^{-rt_2}f_{\tau_2}(t_2)dt_2\right)f_{\tau_1}(t_1)dt_1 \end{aligned} \tag{32}$$

Proof. This follows by substituting properly from Propositions 1 and 2. □

Remark 5. *The "price of reorganizing" for shareholders within Black-Cox framework would be the difference between the value of DBC and BC:*

$$E_0\left[\left(E_{\tau_1}\left[\begin{array}{c} e^{-r(T_2-\tau_1)}K_2 1_{\{\tau_2>T_2\}}1_{\{X_{T_2}>\ln K_2\}}+ \\ e^{-r(T_2-\tau_1)}\exp(X_{T_2})1_{\{\tau_2>T_2\}}1_{\{X_{T_2}\le\ln K_2\}}+e^{-r\tau_2}V_L 1_{\{\tau_2\le T_2\}} \end{array}\right]-V_R\right)1_{\{\tau_1\le T_1\}}\right] \tag{33}$$

4 Numerical results

Once the expressions for pricing debt and equity in the event of a reorganization have been obtained, we provide some numerical results with the aim of clarifying the impact that variables and parameters have for each of the two approches we propose.

First, in Tables 1 and 2 we provide two examples of pricing debt and equity using the four approaches we have studied: Merton (1974) (expressions 11 and 12), Black and Cox (1976) (expressions 13 and 14) and the Single (expressions 16-19 and 22-25) and Double Black-Cox models (expressions 29 and 32) proposed in this paper. In Table 1, the value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The company value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ (70% of company's value) which equals the debt amount at first maturity, that is $K_1 = 0.7$, and $V_L = 0.5$, respectively. The debt amount at second maturity in case of reorganization is $K_2 = 0.7$, which equals K_1. In Table 2, K_2 is smaller than K_1, that is, debtholders offer a reduction in the face value of debt. It can be noticed that in both tables the value obtained for shareholders with Black and Cox (1976) is lower than the value given by Merton (1974). This result is reasonable because there is a higher probability of default in the former structure compared to Merton's approach. Moreover the value obtained by using the Double Black-Cox is lower than the one obtained by using Single Black-Cox. The reasoning for this result is similar, because allowing for defaults before maturity increases the likelihood of defaults and therefore puts more pressure on shareholders. For bondholders the opposite is obtained. We must also notice that the value of equity in the Single and Double Black-Cox approaches is higher than in the Black-Cox approach, that is, the possibility of reorganizing the firm increases the value of equity. This result is consistent with Leland (1994), who obtains that the possibility of debt restructuring increases the value of equity, and with Chi and Tang (2007), who show the potential gains from investing in the equity of firms under reorganization.

Next, we describe the impact that some meaningful parameters, like maturity time, the face value of debt and the thresholds, have on the credit spread of the firm. Figure 1 describes credit spreads versus T_1 for all four approaches. The assumptions are as in Table 1 but with $T_2 = 2T_1$ (this selection is based in common practices and it is used only for easiness of explanation). It is observed that as maturity increases there exists a clear difference between all four approaches, where Single Black-Cox approach gives the biggest credit spread followed by Merton, Double BC and BC. Figure 2 plots the credit spreads versus the ratio K_2/K_1 for all four approaches. The assumptions are as in Table 1. This ratio gives an idea of which method could be more beneficial for debtholders. The Single Black-Cox approach is more risky when the face value of debt decreases after reorganizing, that is, it gives a higher credit spread, while it is as risky as the Double Black-Cox approach when the face value of debt increases after reorganization. In Figure 3 we describe credit spreads versus the distance between maturity times, $T_2 - T_1$, for all four approaches. The assumptions are as in Table 1 but with $T_1 = 1$. The result is similar to those in Figure 1 but more focused on the maturity time after reorganization. Once again Single BC is the more risky approach, leading to important differences as the second maturity time increases. This implies that short waiting times between reorganization and final maturity is optimal for SBC approach.

Finally, in Figures 4 and 5, the impact of the thresholds, V_R and V_L, on the credit spread are shown. The assumptions are as in Table 1. In Figure 4 we plot credit spreads versus V_R for all four approaches. We must remark that $V_R \in (V_L, V_0)$. It can be seen that the Single Black-Cox approach is still the most expensive in these situations, that is, it has the biggest credit spread. Figure 5 describes credit spreads versus V_L for all four approaches. We should notice that $V_L \in (0, V_R)$ and, as expected, three methods are constant, Merton (1974), Black and Cox (1976) and the Single Black-Cox model, that is, there is no influence of V_L.

5 Conclusions

In this paper we have developed two first-passage-time models for pricing risky debt and equity when the firm has the possibility of carrying out a reorganization. These two approaches generalize the Black and Cox (1976) approach, in which default is the only possibility for a firm in case of financial distress. In practice, reorganization consists of an alternative to liquidation as an attempt to preserve the company as an operating concern.

The first model we propose is the Single Black-Cox approach. Within this approach, a reorganization of the firm is started as soon as the value of the firm crosses a lower threshold. We allow for a change in the face value of debt as well as in the time to maturity. This way, we do not impose any restriction on the new face value of debt, that is, it can be bigger, equal or lower than the initial value. This is consistent with Haugen and Senbet (1978, 1988), who argue that debtholders can offer to reduce the debt claim in order to avoid bankruptcy. The Single Black-Cox analysis is more realistic than other first-passage models because it allows for reorganization before the initial maturity, and it does not restrict the characteristics of the new debt.

Second, we propose the Double Black-Cox approach, that consists of a generalization of the Single Black-Cox approach in the sense that after reorganization the default may occur at anytime before the new maturity time. In this model, we also allow for changes in the face value of debt and in time to maturity. This seems to be a more realistic approach, because it allows default as soon as the value of the assets of the firm is too low, that is, the firm can default without waiting until the new maturity time.

Finally, we provide several numerical results with the aim of clarifying the impact that variables and approaches have for each of the approaches studied. We show that within the framework we propose, the value of equity is higher when there exists the possibility of a reorganization. This result is consistent with the evidence supported by other authors, as Leland (1994) and Chi and Tang (2007). Furthermore, we find that Black and Cox (1976) and Double Black-Cox approaches are beneficial for bondholders in the sense that give protection against large losses, while they are detrimental for shareholders. On the other hand, we observe that Merton (1974) and Single Black-Cox settings could be more risky to bondholders because of the probability of the values of the company to be very small.

References

Abínzano, I., Navas, J. F., 2007, "Pricing the equity of a firm using extendible options", *Revista de Economía Financiera* (forthcoming).

Altman, E., 1984, "A further empirical investigation of the bankruptcy cost question", *Journal of Finance*, 39, 1067 - 1089.

Bebchuk, L. A., 1988, "A new approach to corporate reorganizations", *Harvard Law Review*, 101, 4, 775 - 804.

Bergman, Y. Z., Callen, J. L., 1991, "Opportunistic underinvestment in debt renegotiation and capital structure", *Journal of Financial Economics*, 29, 137 - 171.

Black, F., Cox, J. C., 1976, "Valuing corporate securities: Some effects of bond indenture provisions", *Journal of Finance*, 31, 351 - 367.

Branch, B., 2002, "The costs of bankruptcy. A review", *International Review of Financial Analysis*, 11, 39-57.

Brown, D.T., 1989, "Claimholder incentive conflicts in reorganization: The role of bankruptcy law", *Review of Financial Studies*, 2, 1, 109 - 123.

Chatterjee, S., Dhillon, U.S., Ramírez, G.G., 1995, "Coercive tender and exchange offers in high yield debt restructurings: An empirical analysis", *Journal of Financial Economics*, 333-360.

Chatterjee, S., Dhillon, U.S., Ramírez, G.G., 1996, "Resolution of financial distress: Debt restructurings via Chapter 11, prepackaged bankruptcies, and workouts", *Financial Management*, 25, 1, 5-18.

Chi, L, Tang, T., 2007, "Impact of reorganization announcements on distressed-stock returns", *Economic Modelling*, 24, 749-767.

Dumitrescu, A., 2007, "Valuation of defaultable bonds and debt restructuring", *Journal of Corporate Finance*, 13, 94-111.

Ericson, J., Reneby, J., 1998, "A framework for valuing corporate securities", *Applied Mathematical Finance*, 5, 143 - 163.

Fan, H., Sundaresan, S. M., 2000, “Debt valuation, renegotiation, and optimal dividend policy”, *Review of Financial Studies*, 13, 1057 - 1099.

Fisher, T., Martel, J., 2003, “The effects of Bankruptcy Reform on the number of corporate reorganization proposals”, *Canadian Public Policy*, 29, 3, 339-350.

Francois, P., Morellec, E., 2004, “Capital structure and asset prices: Some effects of bankruptcy”, *Journal of Business*, 77, 387- 411.

Franks, J., Torous, W., 1989, “An empirical investigation of U.S. firms in reorganization”, *Journal of Finance*, 44, 747 - 769.

Galai, D., Raviv, A., Wiener, Z., 2007, “Liquidation triggers and the valuation of equity and debt”, *Journal of Banking and Finance*, 31, 3604-3620.

Gertner, R., Scharfstein, D., 1991, “A theory of workouts and the effects of Reorganization Law”, *Journal of Finance*, 1189-1222.

Haugen, R., Senbet, L., 1978, “The insignificance of bankruptcy costs to the Theory of Optimal Capital Structure”, *Journal of Finance*, 33, 383-393.

Haugen, R., Senbet, L., 1988, “Bankruptcy and agency costs: Their significance to the Theory of Optimal Capital Structure”, *Journal of Financial and Quantitative Analysis*, 23, 27-38.

He, H., Keirstead, W., Rebholz, J., 1998, “Double Lookbacks”, *Mathematical Finance*, 8, 3, 201-228.

Leland, H., 1994, “Corporate debt value, bond covenants, and optimal capital structure”, *Journal of Finance*, 49, 4, 1213 - 1252.

Longstaff, F., 1990, “Pricing options with extendible maturities: Analysis and applications”, *Journal of Finance*, 45, 935 - 957.

Merton, R. C., 1974, “On the pricing of corporate debt: The risk structure of interest rate” , *Journal of Finance*, 29, 449 - 470.

Warner, J., 1977, “Bankruptcy costs: Some evidence”, *Journal of Finance*, 32, 2, 337-347.

	Equity	**Debt**
Merton(V_0, V_R, K_1, T_1)	0.3574	0.6426
Black-Cox(V_0, V_R, K_1, T_1)	0.3000	0.7000
$SBC(V_0, V_R, K_1, K_2, T_1, T_2)$	0.3967	0.6033
$DBC(V_0, V_R, V_L, K_1, K_2, T_1, T_2)$	0.3792	0.6208

Table 1: Pricing of debt and equity using four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$.

	Equity	**Debt**
Merton(V_0, V_R, K_1, T_1)	0.3574	0.6426
Black-Cox(V_0, V_R, K_1, T_1)	0.3000	0.7000
$SBC(V_0, V_R, K_1, K_2, T_1, T_2)$	0.4520	0.5480
$DBC(V_0, V_R, V_L, K_1, K_2, T_1, T_2)$	0.4240	0.5760

Table 2: Pricing of debt and equity using four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.5$.

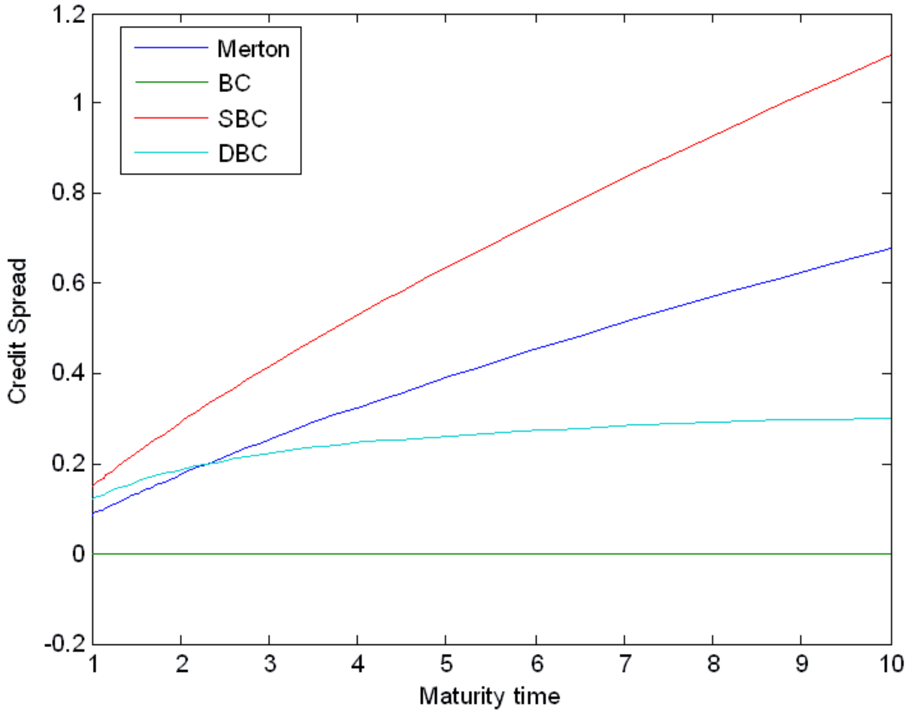

Figure 1: Effect of Maturity time on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The second maturity date is $T_2 = 2T_1$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$.

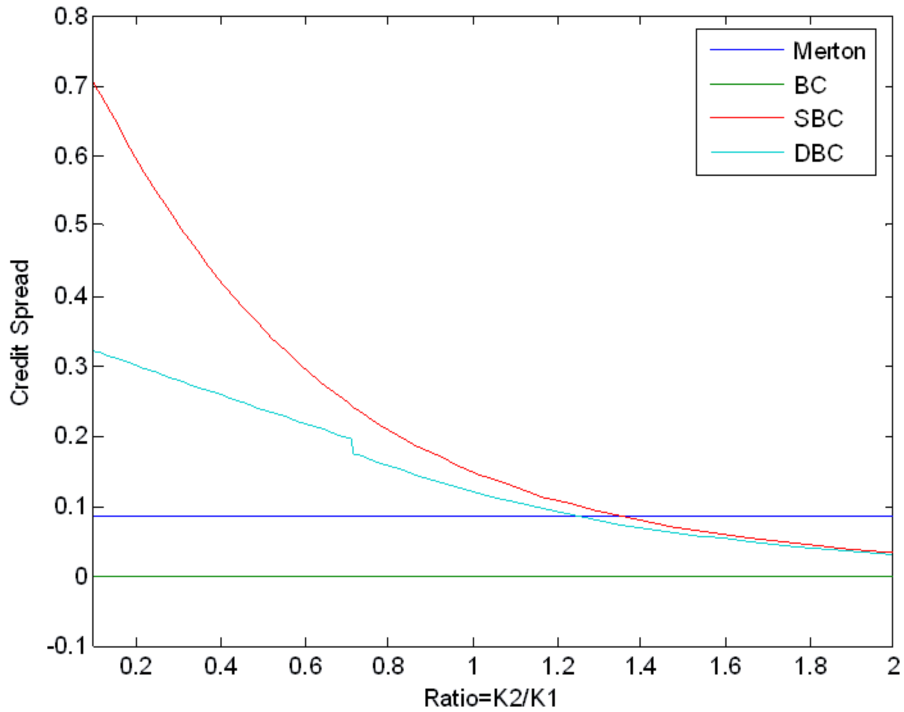

Figure 2: Influence of the ratio K_2/K_1 on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively.

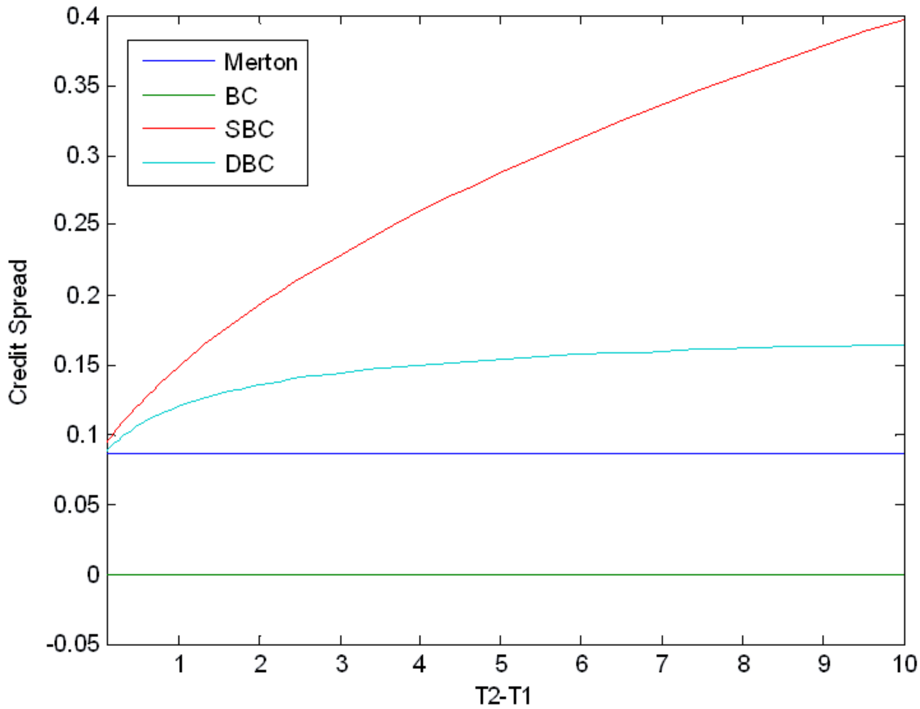

Figure 3: Influence of $T_2 - T_1$ on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$. The thresholds for reorganization and for liquidation are $V_R = 0.7$ and $V_L = 0.5$, respectively. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$.

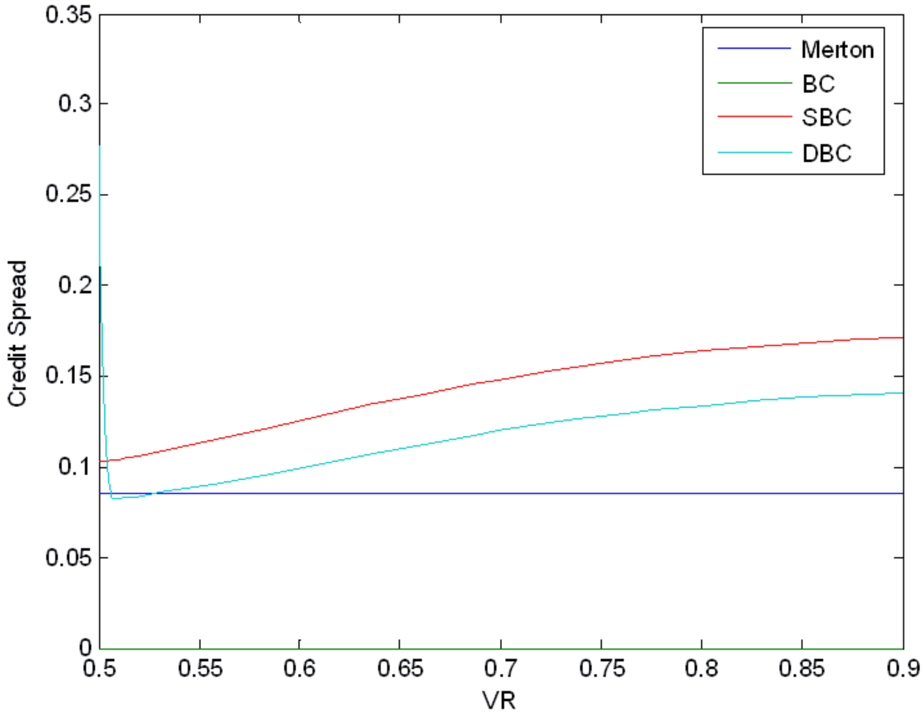

Figure 4: Influence of V_R on the credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The threshold for liquidation is $V_L = 0.5$. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$.

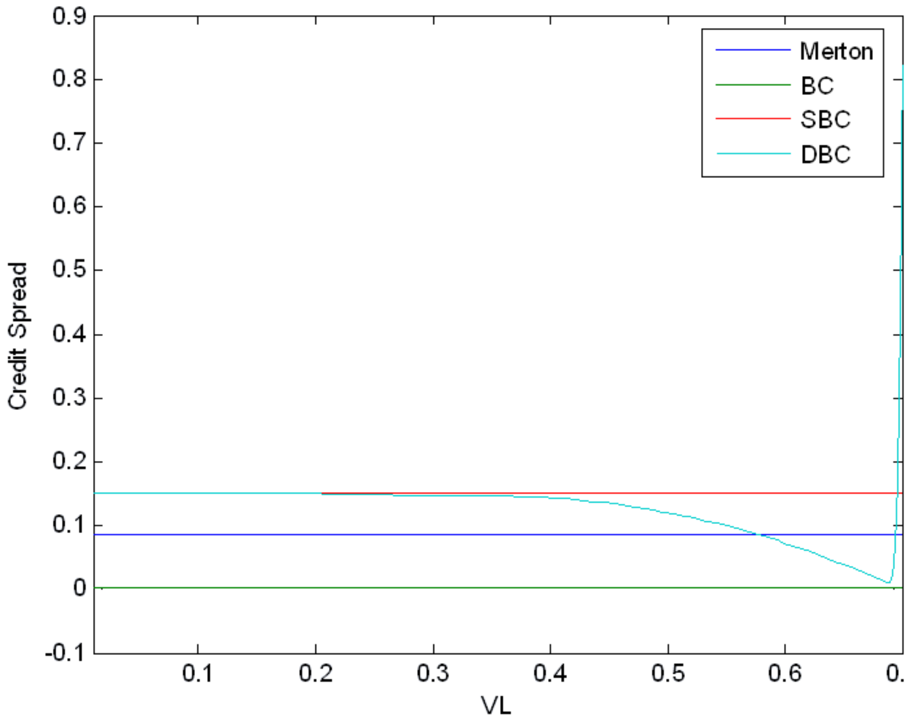

Figure 5: Influence of V_L on the Credit spread for the following four approaches: Merton (1974), Black and Cox (1976) and the Single and Double Black-Cox models proposed in this paper. The value of the volatility is $\sigma = 0.5$ and the risk-free interest rate is $r = 0$. The assets value at zero is $V_0 = 1$. The first maturity date is $T_1 = 1$ and the second maturity date, in case of reorganization, is $T_2 = 2$. The threshold for reorganization is $V_R = 0.7$. The debt amount at first maturity is $K_1 = 0.7$ and at second maturity is $K_2 = 0.7$.