# UNCERTAINTY AND THE PRICE OF RISK IN A NOMINAL CONVERGENCE PROCESS

## ( VERY PRELIMINAR AND INCOMPLETE DRAFT)

Ricardo Gimeno Nogués

BANCO DE ESPAÑA

José Manuel Marqués Sevillano

BANCO DE ESPAÑA

## Abstract

Standard approaches to measure the evolution of real interest rate in Spain points out a sharp decrease of this variable in any term of the yield curve. However, as Blanco and Restoy (2006) shows, it's difficult to explain this evolution based on the recent behaviour of Spanish macro figures. In this paper we estimate during the 90's the real interest rate and the risk premia using an affine model based and the non arbitrage condition along the yield curve. According to recent literature we use alternative set of explanatory variables to obtain the decomposition of nominal interest rate. The model based on the Nelson–Siegel exponential components of the yield curve produces the most robust results. The evidence indicates that inflation expectations and risk premia account for most of the observed variation in nominal rates while real interest rate shows only an smooth reduction during this period.

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# 1 INTRODUCTION

Describing the joint behaviour of real interest rate and inflation expectation is important for bond pricing, investment decisions and monetary policy. However both variables are unobserved and the large empirical literature has yield few accepted stylized facts. In fact we only can observe directly the nominal interest rate and the ex-post inflation rate. Therefore to obtain an approach to real interest rate some assumptions has to be made for inflation expectations and the risk premia. From a macroeconomic perspective is usual to impose constant risk premia in order to focus on inflation expectations. However from a finance perspective risk premia is the key variable in order to pricing assets for several maturities. This task is specially difficult in a nominal convergence process similar to the one observed in the nineties for several countries involved in the European monetary union process or other countries that entered in a long period of stabilization, where shifts are expected in the inflation process

In this regard our work is related to papers like Diebold, Rudebuch and Boragan (2004) Carriero, Favero and Kaminska (2006) or Ang, Bekaert and Wei (2006) (ABW) that incorporates macro determinants into a multi-factor yield curve models with non arbitrage opportunities. This macro-finance approach allows combining a good assessment of inflation expectations with a realistic dynamic for risk premia and real interest rates.

We considered a model where interest rates are affine with a vector of factors that includes the inflation rate. Moreover the non-arbitrage opportunities condition should be satisfied along the yield curve. Taken together this two condition is straightforward to decomposed nominal interest rates as the addition of real risk-free interest rates, expected inflation an the risk premia. The last variable could be derived either as part of the inflation compensation, such as ABW (2006) or simply by imposing in the valuation of the nominal price of bond that the price of risk should be equal to zero, similar to Ang and Piazzesi (2005).

Two alternatives are considered as factors other than inflation. One alternative, that was intensively used in the literature in papers like Dai and Singleton (2000), Laubach and Williams (2003) or ABW (2006), consist in using unobserved components that are estimated by a Kalman Filter. However, this approach heavily depends on the initial conditions and is difficult to obtain a parsimonious result. Some papers surround this inconvenient by imposing several restrictions to the parameters that in some cases are not fully justified. In this paper we deal with the initial condition problem by using genetic algorithm in the estimation procedure. The other alternative was based on the Nelson–Siegel exponential components of the yield curve (Nelson and Siegel, 1987) in a similar vein to Carriero, Favero and Kaminska (2004) or Diebold and Li (2006). This framework did not depend on initial parameters and was based only on Maximum likelihood estimation. Moreover, in both cases, the dynamic of the vector factor was modelled as a VAR structure, like Diebold, Rudebusch and Boragan (2004) proposes in order to allow for the interaction between inflation and the rest of components that could be reflecting variables like natural interest rate, potential growth of the economy or risk aversion.

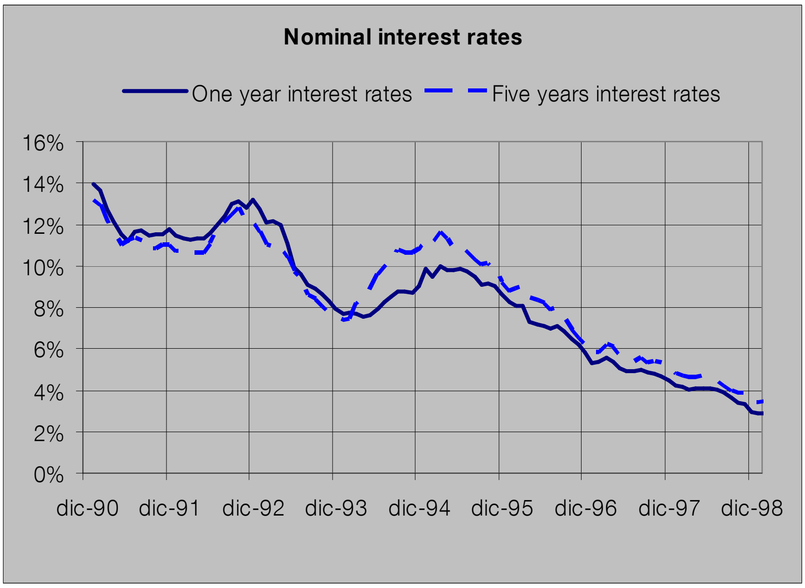
This paper was conducted with Spanish data for the period previous to the European Monetary Union. As Blanco and Restoy (2006) points out during this period Spanish economy has experience a dramatically transformation that makes difficult the decomposition of nominal interest rate. Moreover, during this period the economy can be affected by a peso problem related with the entry on EMU that was not completely capture by other macroeconomic and financial approaches. In order to take account of this peso problem we include a Markov switching regime for the dynamic of the factors. However our results show that this procedure is not really necessary to capture the switching regime.

The rest of the paper was structured on four additional sections. In the second section a description of nominal interest rate and inflation in Spain was presented. In the third section we present the basic model and derive the decomposition of nominal interest rate. The fourth section presents and asses the main results with the different approaches. Finally Section 5 concludes.

## 2 INTEREST RATE EVOLUTION IN SPAIN

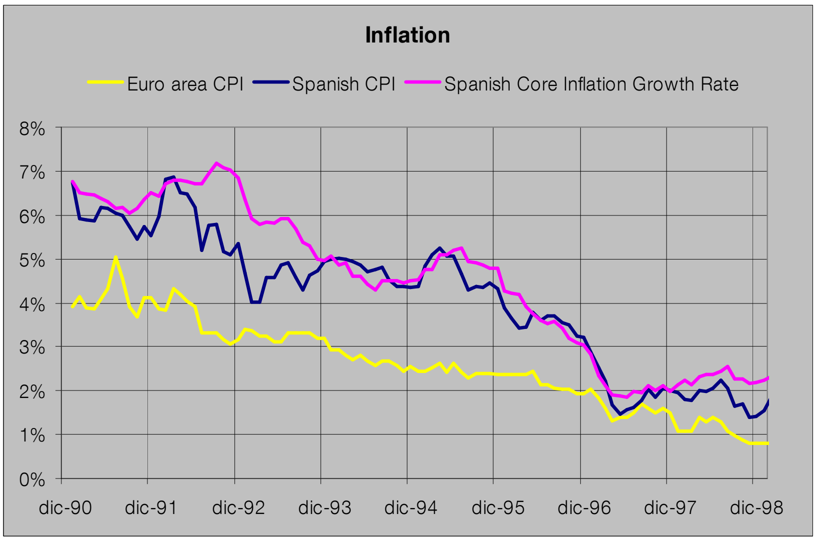
In Spain, Interest rate evolution in the nineties was mainly driven by a nominal convergence process according to the expectancies on entering the European Monetary Union (EMU). This process produces a reduction on interest rates never previously seen in the whole record of Spanish data. As can be seen in Figure 1, interest rates where at the beginning of the decade at levels around 12%, closing the nineties in levels that goes from 3% to 4% depending on the term considered.

**Figure 1: Nominal interest rates evolution in the nineties**



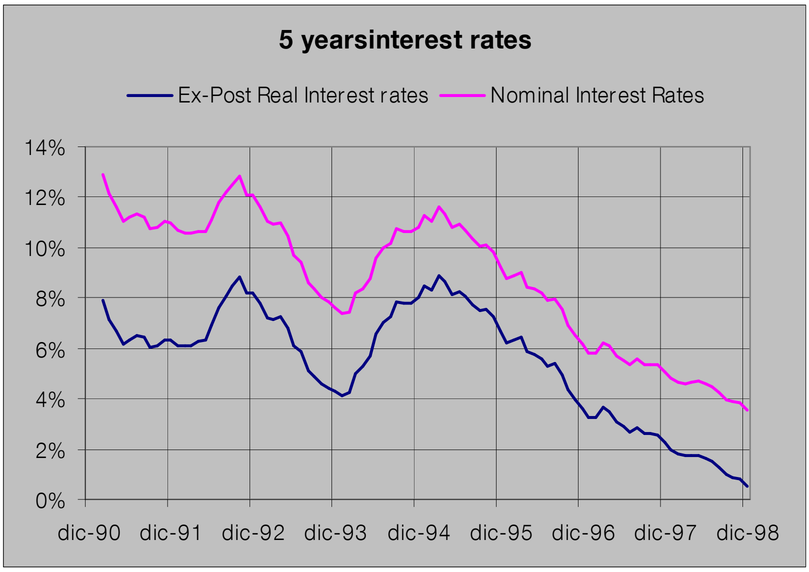
A similar convergence process was reached in terms of inflation, as can be seen in figure 2. Both European and Spanish CPI experienced a reduction in levels. Main differences between both figures may be found on the first part of the decade. While volatility was quite bigger for Spain, the reduction was not so obvious for highlight inflation data until 1996, when the Spanish peseta increase its chances of being included on the Euro.

**Figure 2: Spanish and European Consumer Price Indexes**



In fact, the reduction of observed inflation is not able to explain the whole reduction of nominal interest rates. As can be seen in figure 3, if we subtract observed inflation from nominal interest rates, to obtain ex-post real interest rates, the reduction would be of around 7 p.p. in less than 4 years.

**Figure 3: 5-years nominal and ex-post real interest rates**



As Blanco and Restoy (2007) mention, Spanish economy did not seems to have suffered the impact that a reduction of real interest rates of this magnitude would have implied. So, maybe there is an alternative explanation for the evolution of nominal interest rates. The certainty over the Spanish incorporation to the EMU was not clear all around the nineties, and at some moments the process was in serious danger. This must have implied some consequence, both in inflation expectations and risk premium that, given the final solution, were not materialized.

As a consequence, analyzing these magnitudes in depth would give us some information relevant to a better comprehension of the nominal convergence process of the economies involved on the EMU project.

## 3 Real rates estimation

### 3.1 The affin model

A risk-free zero coupon bond paying 1 unit in period *t+k* could be valued in period *t* according to the classical financial valuation function on continuous time as,

$$P_t^k = e^{-ky_{t,t+k}} \tag{1}$$

where $y_{t,t+k}$ is the nominal interest rate between period *t* and *t+k*. The affine model assumes that you can rearrange this valuation function using some affine (linear) factors ($X_t$),

$$P_t^k = e^{A_k + B_k' X_t} \tag{2}$$

This equation assumes that nominal yields are completely determined by these affine factors,

$$y_{t,t+k} = -\frac{1}{k}\left(A_k + B_k X_t\right) \tag{3}$$

Note that differences in interest rates along the term structure will be consequence of different coefficients $A_k$ and $B_k'$, whereas changes along time will be driven by movements in the affine factors. These changes in the affine factors are supposed to follow dynamics that can be approach by a VAR model,

$$X_t = \mu + \Phi X_{t-1} + \varepsilon_t \tag{4}$$

where $\mu$ is a vector of the constant drift in the affine variables $X_t$, while $\Phi$ is a matrix of the autoregressive coefficients. Finally $\varepsilon_t$ is a vector of random variables that follows a Gaussian normal distribution, $\varepsilon_t = \Sigma v_t$, $v_t \sim N(0,I)$. The VAR model accounts for the observed predictability in the interest rates but allowing at the same time for some degree of uncertainty in the future values of interest rates, represented by perturbation variables $\varepsilon_t$. These random variables are a source of risk with consequences along the term structure.

#### 3.1.1 THE RISK PREMIUM

To incorporate the uncertainty in the valuation framework we could consider that future payouts are not known (implying some type of default risk or similar) or future interest rates are not known (term risk). Given that sources of uncertainty, the actual value of an asset would satisfies the following equation:

$$P_t = E_t^Q\left[e^{-y_{t,t+1}}\right] \tag{5}$$

That states that discounting with the risk free rate ($y_{t,t+1}$) we obtain risk neutral measures. In order to obtain non risk neutral measures we use the Radon-Nikodym derivative (which converts the risk neutral measure), denoted by $\xi_t$. So it is possible to pass from a risk neutral world to a risk averse one.

$$E_t^Q\left[e^{-r_t}\right] = E_t\left[\xi_{t+1} e^{-y_{t,t+1}}\right] \Big/ \xi_t \qquad (6)$$

Supposing $\xi_t$ follows a log-normal process,

$$\xi_{t+1} = \xi_t e^{\left(-\frac{1}{2}\lambda_t{}'\lambda_t - \lambda_t{}'\varepsilon_{t+1}\right)} \qquad (7)$$

where $\lambda_t$ is the price of risk that multiplies the perturbation vector $\varepsilon_{t+1}$. The first part of the exponent ( $\lambda_t{}'\lambda_t$ ) is the Jensen Convexity factor. Substituting (2) in (1),

$$E_t^Q[e^{-y_{t,t+1}}] = E_t\left[\xi_t e^{\left(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t{}'\varepsilon_{t+1}\right)} e^{-y_{t,t+1}}\right] \Big/ \xi_t = E_t\left[e^{\left(-y_{t,t+1}-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t{}'\varepsilon_{t+1}\right)}\right] \qquad (8)$$

where $e^{\left(-y_{t,t+1}-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t{}'\varepsilon_{t+1}\right)} = m_{t+1}$ is the pricing kernel. The price of risk could be modelized assuming to be also affine to factors $X_t$,

$$\lambda_t = \lambda_0 + \lambda_1 X_t \qquad (9)$$

Finally we reach to expression,

$$P_t = E_t\,[m_{t+1}] \qquad (10)$$

In the case of a risk-free zero coupon bond (assuming no default risk), then,

$$P_t = E_t\left[m_{t+1}\right] = E_t\left[m_{t+1}\right] = E_t\left[e^{\left(-y_{t,t+1}-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t{}'\varepsilon_{t+1}\right)}\right] = e^{-y_{t,t+1}-\frac{1}{2}\lambda_t'\lambda_t} E_t\left[e^{-\lambda_t{}'\varepsilon_{t+1}}\right] \qquad (12)$$

where the expectation operator on a nonlinear function of a random variable implies the appearance again of a Jensen Inequality component, $E_t\left[e^{-\lambda_t{}'\varepsilon_{t+1}}\right] = e^{\frac{1}{2}\lambda_t{}'\lambda_t}$, so $P_t$ will be,

$$P_t = e^{-y_{t,t+1}} \qquad (13)$$

implying that one period estimation includes no risk premium. That will not be the case for multi-period interest rates.

### 3.1.2 THE RECURSIVE TERM STRUCTURE

Now we are able to use recursive computation of parameters. Assuming for a zero coupon bond paying 1 unit in period *t+k* you can price it linearly to the affine factors,

$$P_t^k = e^{A_k + B_k' X_t} \qquad (14)$$

So, for one period bond,

$$P_t^1 = E_t\left[m_{t+1}\right] = e^{-y_{t,t+1}} = e^{A_1 + B_1' X_t} = e^{-\delta_0 - \delta_1' X_t} \tag{15}$$

that defines the affine structure of one-period nominal interest rates in terms of the affine factors $X_t$, a constant $\delta_0$ and a vector of coefficients $\delta_1'$.

$$y_{t,t+1} = \delta_0 + \delta_1' X_t \tag{16}$$

For other periods you could use a recursive function. For example, evaluation of the price in *t* of an asset with term in *k*+1, we just need its price in *t*+1 to discount it using one period discount kernel $m_{t+1}$,

$$P_t^{k+1} = E_t\left[m_{t+1} p_{t+1}^k\right] = E_t\left[m_{t+1} e^{A_k + B_k' X_{t+1}}\right] = E_t\left[e^{-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}} e^{A_k + B_k' X_{t+1}}\right] \tag{17}$$

The evaluation of the expectation operator is not trivial and requires some previous manipulation on equation (17). First of all, we extract all part of the exponentials known on *t* (not affected by the expectation operator).

$$P_t^{k+1} = \cdots = e^{-\delta_0 - \delta_1{}' X_t - \frac{1}{2}\lambda_t'\lambda_t + A_k} E_t\left[e^{-\lambda_t'\varepsilon_{t+1} + B_k' X_{t+1}}\right] \tag{18}$$

On equation (18), only the perturbation vector $\varepsilon_{t+1}$ and the future value of affine factors $X_{t+1}$ are not known on *t*. Now we take advantage of the VAR equation (4) to substituting $X_{t+1}$,

$$\begin{aligned} P_t^{k+1} &= e^{-\delta_0 - \delta_1{}' X_t - \frac{1}{2}\lambda_t'\lambda_t + A_k} E_t\left[e^{-\lambda_t'\varepsilon_{t+1} + B_k'(\mu + \Phi X_t + \Sigma\varepsilon_{t+1})}\right] = \\ &= e^{-\delta_0 - \delta_1{}' X_t - \frac{1}{2}\lambda_t'\lambda_t + A_k + B_k'(\mu + \Phi X_t)} E_t\left[e^{-\lambda_t'\varepsilon_{t+1} + B_k'\Sigma\varepsilon_{t+1}}\right] \end{aligned} \tag{19}$$

Now only the perturbation vector $\varepsilon_{t+1}$ remains in the expectation operator of equation (19), that will be equal to,

$$E_t\left[e^{-\lambda_t'\varepsilon_{t+1} + B_k'\Sigma\varepsilon_{t+1}}\right] = e^{\frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}B_k'\Sigma\Sigma' B_k' - B_k'\Sigma\lambda_t} = e^{\frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}B_k'\Sigma\Sigma' B_k' - B_k'\Sigma(\lambda_0 + \lambda_1 X_t)} \tag{20}$$

The first two components in this formula (equation 20) come from the Jensen inequality ($\frac{1}{2}\lambda_t'\lambda_t$, $\frac{1}{2}B_k'\Sigma\Sigma' B_k'$) while the third one is consequence of the covariance between $-\lambda_t'\varepsilon_{t+1}$ and $B_k'\Sigma\varepsilon_{t+1}$. Finally, putting all together we get,

$$\begin{aligned} P_t^{k+1} &= e^{-\delta_0 - \delta_1{}' X_t - \frac{1}{2}\lambda_t'\lambda_t + A_k + B_k'(\mu + \Phi X_t) + \frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}B_k'\Sigma\Sigma' B_k' - B_k'\Sigma(\lambda_0 + \lambda_1 X_t)} = \\ &= e^{-\delta_0 + A_k + B_k'\mu - B_k'\Sigma\lambda_0 + \frac{1}{2}B_k'\Sigma\Sigma' B_k' + (-\delta_1' + B_k'\Phi - B_k'\Sigma\lambda_1)X_t} = e^{A_{k+1} + B_{k+1}' X_t} \end{aligned} \tag{21}$$

So the coefficients $A_k$ and $B_k'$ in the affine representation will be,

$$A_{k+1} = -\delta_0 + A_k + B_k'\mu - B_k'\Sigma\lambda_0 + \frac{1}{2}B_k'\Sigma\Sigma'B_k' \tag{22}$$

$$B_{k+1}' = -\delta_1' + B_k'\Phi - B_k'\Sigma\lambda_1 \tag{23}$$

### 3.1.3 DECOMPOSITION OF NOMINAL INTEREST RATES

From previous section we have defined that nominal interest rates will be equal to,

$$y_{t,t+k} = \frac{-1}{k}\left(A_k + B_k'X_t\right) \tag{24}$$

where $X_t$ are the factor variables, while $A_k$ and $B_k$ are parameters obtained from the recursive method of equations (22) and (23). $k$-periods nominal interest rates could be decomposed in real rates ($r_{t,t+k}$), inflation expectations ($E_t\left[\pi_{t,t+k}\right]$) and risk premium ($\gamma_{t,t+k}$).

$$y_{t,t+k} = r_{t,t+k} + E_t\left[\pi_{t,t+k}\right] + \gamma_{t,t+k} \tag{25}$$

Real rates are obtained subtracting from the nominal interest rates inflation expectations and inflation risk premium. Inflation expectations could be recovered from VAR equation (4) of the affine factors, where $\pi_{t,t+1}$ was the last component of vector $X_t$.

$$E_t\left[X_{t+h}\right] = \left(1 + \Phi + \Phi^2 + \cdots + \Phi^{h-1}\right)\mu + \Phi^h X_t \tag{26}$$

From previous equation we obtain successive $E_t\left[\pi_{t+h,t+h+1}\right]$ that must be integrated to obtain $E_t\left[\pi_{t,t+k}\right]$.

$$E_t\left[\pi_{t,t+k}\right] = \prod_{h=0}^{k-1}\left(1 + E_t\left[\pi_{t+h,t+h+1}\right]\right) - 1 \tag{27}$$

As stated in section 1.2, risk premium appears as a consequence of the existence of uncertainty in the future value of the affine factors driven by perturbation $\varepsilon_t$ of the VAR equation. Price of risk $\lambda_t$ was the time-varying parameter that incorporate risk premium to the term parameter, so by imposing $\lambda_t = 0$ to the computation of parameters $A_k$ and $B_k$ (equations 22 and 23) we will subtract the risk premium from the term structure (Ang and Piazzesi, 2004).

$$\gamma_{t,t+k} = y_{t,t+k} - y\left(\lambda_0 = 0, \lambda_1 = \bar{0}\right)_{t,t+k} \tag{28}$$

The presence of nonlinearities in the expectation operator also includes an additional component in the term structure, the Jensen inequality component. This component affects the computation of $A_{k+1}$ by adding $\frac{1}{2}B_k'\Sigma\Sigma'B_k'$ to the recursive function. We can also modify the risk premium in order to get rid of this component, simply by getting rid of the noise in the VAR equation, or adding the restriction $\Sigma = 0$ to the computation of parameters $A_k$ and $B_k$,

$$\begin{aligned}\gamma^{*}{}_{t,t+k} &= \left(y_{t,t+k} - y(\lambda_t = 0)_{t,t+k}\right) + \left(y(\lambda_t = 0)_{t,t+k} - y\left(\Sigma = \overline{0}\right)_{t,t+k}\right) = \\ &\quad y_{t,t+k} - y\left(\Sigma = \overline{0}\right)_{t,t+k}\end{aligned} \tag{29}$$

where,

$$y\left(\Sigma = \overline{0}\right)_{t,t+k} = \frac{-1}{k}\left(A^{*}{}_{k} + B'^{*}{}_{k} X_t\right) \tag{30}$$

being coefficients $A_k^{*}$ and $B_k^{*\prime}$ similar to the ones obtained in the recursive style of equation (28), but without the terms with matrix $\Sigma$

$$A_{k+1} = -\delta_0 + A_k + B_k'\mu \tag{31}$$

$$B_{k+1}' = -\delta_1' + B_k'\Phi \tag{32}$$

Once both inflation expectations and risk premium have been computed, real rates could be estimated,

$$r_{t,t+k} = y_{t,t+k} - E_t\left[\pi_{t,t+k}\right] - \gamma^{*}{}_{t,t+k} \tag{33}$$

### 3.1.4 ABW DECOMPOSITION

The decomposition of equation (33) assumes that real interest rates are stripped from any risk premium, while variations along the term structure being consequence of the expected variations in the factors $X_t$. A different approach to the decomposition of real interest rates is the one Proposed by Ang, Bekaert and Wei (2006), that we will call ABW decomposition. They conserve part of the risk premium in the real rates subtracting from nominal rates what they called inflation compensation that includes the inflation expectation and a risk premium directly linked to inflation uncertainty.

So, in the first period real rates could be obtained from nominal rates simply by subtracting expected inflation.

$$\begin{aligned} r_{t,t+1} &= y_{t,t+1} - E\left[\pi_{t+1}\right] = \delta_0 + \delta_1 X_t - E\left[e_N X_{t+1}\right] = \\ &= \delta_0 + \delta_1 X_t - e_N E\left[\mu + \Phi X_t + \varepsilon_{t+1}\right] = \\ &= \delta_0 X_t - e_N\left(\mu + +\Phi X_t\right) = \delta_0 - e_N\mu + \left(\delta_1 - e_N\Phi\right)X_t \end{aligned} \tag{34}$$

so the real pricing kernel will be now,

$$p_t^1 = E_t\left[\tilde{m}_{t+1}\right] = e^{-r_{t,t+1}} = e^{\tilde{A}_1 + \tilde{B}_1' X_t} = e^{-\delta_0 + e_N\mu - \left(\delta_1' - e_N\Phi\right)X_t} \tag{35}$$

whereas for other periods we will have,

$$r_{t,t+k} = -\frac{1}{k}\left(\tilde{A}_k + \tilde{B}_k' X_t\right) \tag{36}$$

similar to the recursive structure of nominal interest rates, we could obtain coefficients $\tilde{A}_{k+1}$ and $\tilde{B}_{k+1}'$ simply applying the non arbitrage condition,

$$p_t^{k+1} = E_t\left[\tilde{m}_{t+1} p_{t+1}^{k}\right] = E_t\left[\tilde{m}_{t+1} e^{\tilde{A}_k + \tilde{B}_k' X_{t+1}}\right] = \\ = E_t\left[e^{-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}} e^{\tilde{A}_k + \tilde{B}_k' X_{t+1}}\right] \tag{37}$$

this is similar to the evaluation of the nominal equation. If we take out of the expectations the constant coefficients we would obtain,

$$P_t^{k+1} = \cdots = e^{-\delta_0 + e_N\mu - \left(\delta_1' - e_N\Phi\right)X_t - \frac{1}{2}\lambda_t'\lambda_t + \tilde{A}_k} E_t\left[e^{-\lambda_t'\varepsilon_{t+1} + B_k' X_{t+1}}\right] \tag{38}$$

### 3.2 THE FACTORS ON THE AFFINE MODEL

Affine models usually assume that the factor models are not observable so a Kalman filter must be used in order to recover them. Kalman filter impose some extra problems to the estimation of parameters in the model. First of all, the extra values we have to estimate, implies problems of identification. Also, in order to proceed to the Kalman filter, some of the yields must be observed without error, and the results are quite sensible to the ones we chose to such purpose.

An alternative approach implies the previous determination of the factors affecting the shape of the term structure. That is the case of Diebold and Li (2006), that uses the Nelson and Siegel (1987) term structure specification,

$$y_{t,t+k} = L_t + SL_t \frac{1 - e^{-k/\tau}}{k/\tau} + C_t\left(\frac{1 - e^{-k/\tau}}{k/\tau} - e^{-k/\tau}\right) \tag{39}$$

In order to linealize the estimation, we can fix $\tau$ in the mean value observed along the original sample[1]. So in this way be obtain $L_t$, $SL_t$ and $C_t$ as parameters of the regression of X, where X is a matrix in which each row $k$ is formed by,

$$X_k = \begin{bmatrix} 1 & \frac{1 - e^{-k/\tau}}{k/\tau} & \frac{1 - e^{-k/\tau}}{k/\tau} - e^{-k/\tau} \end{bmatrix} \tag{40}$$

Successive regressions in each period give us the time series of parameters $L_t$, $SL_t$ and $C_t$ that can be considered as factors determining the term structure of interest rates. $L_t$, is the long term interest rate (both forward and spot), $SL_t$ is the spread (difference between long term and sort term interest), while $C_t$ is a measure of the curvature of the term structure.

So if we considered a typical affine model,

$$y_{t,t+k} = \frac{-1}{k}\left(A_k + B_k' X_t\right) + u_{t,t+k} \qquad u_t \sim N\left(0, \sigma^2 I\right) \tag{41}$$

We can substitute the factors,

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[1] Trying to estimate $\tau$ jointly with $L_t$, $SL_t$ and $C_t$ produced non trivial problems of identification as shown by Gimeno and Nave (2006).

$$X_t = \begin{bmatrix} L_t \\ SL_t \\ C_t \end{bmatrix} \tag{42}$$

while the coefficients will be,

$$B'_k = \begin{bmatrix} -k & -\dfrac{1-e^{-k/\tau}}{1/\tau} & -\dfrac{1-e^{-k/\tau}}{1/\tau} + ke^{-k/\tau} \end{bmatrix} \quad \text{and} \quad A_k = 0 \tag{43}$$

Given the predictability of these factors, a VAR(1) model can be fitted to the data.

$$X_t = \mu + \Phi X_{t-1} + \varepsilon_t \tag{44}$$

This equation implies a dynamic in the factors. Nevertheless, this specification of $A_k$ and $B_k$ do not always produce valuations without arbitrage opportunities, as is the case of the procedure of section 1.3. In any case, latent factors usually estimated are closely related with the level and spread of the term structure, so at least $L_t$ and $SL_t$ could be good candidates to affine factors, leaving the determination of the term structure parameters of $A_k$ and $B_k$ to a maximum likelihood estimation.

In order to estimate real rates it is convenient to include inflation among the affine factors. In fact, as suggested by Farriero et al. (2006) the adding of macroeconomic variables[2] to the affine model can actually improve the model. This is exactly what ABW (2006) do.

$$X_t = \begin{bmatrix} L_t \\ SL_t \\ C_t \\ \pi_t \end{bmatrix} \tag{45}$$

To sum up, we will estimate via maximum likelihood the following model:

$$X_t = \mu + \Phi X_{t-1} + \varepsilon_t \qquad \varepsilon_t = \Sigma v_t\,,\ v_t \sim N(0,I) \tag{46}$$

$$y_{t,t+k} = \frac{-1}{k}\left(A_k + B'_k X_t\right) + u_{t,t+k} \qquad u_t \sim N\left(0,\sigma^2 I\right) \tag{47}$$

$$A_{k+1} = -\delta_0 + A_k + B'_k \mu - B'_k \Sigma \lambda_0 + \frac{1}{2} B'_k \Sigma \Sigma' B'_k \tag{48}$$

$$B'_{k+1} = -\delta'_1 + B'_k \Phi - B'_k \Sigma \lambda_1 \tag{49}$$

$$y_{t,t+1} = \delta_0 + \delta'_1 X_t \qquad \lambda_t = \lambda_0 + \lambda_1 X_t \tag{50}$$

### *3.3 MARKOV SWITCHING IN THE AFFINE MODEL*

The process of convergence to the euro, that was common among the actual euro-zone members, imposes the possibility of the existence of the separate regimes among the expected horizon of these countries. The presence of such eventually will implies different

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[2] We introduce CPI inflation on the model, but Taylor rule would suggest that we also incorporate the output gap to the model.

inflation expectations and risk premiums. Such circumstances could be included in the above model.

Let $S_t$ be a Markov chain that presence to possible values identifying the two regimes in the model. Under one regime ($S_t = 1$), low inflation expectations and price of risk are expected, whereas in the other regime, ($S_t = 2$), higher values have more chances of been found. That could be rearrange in the model as,

$$X_t = \mu(S_t) + \Phi X_{t-1} + \varepsilon_t \qquad \varepsilon_t = \Sigma v_t\,,\ v_t \sim N(0,I) \tag{51}$$

where $\mu_\pi(S_t)$ is a regime switching drift of inflation. In the case of the price of risk we include similar coefficients:

$$\lambda_t = \lambda_0(S_t) + \lambda_1 X_t \tag{52}$$

allowing for a different evaluation of the risk implied by random variable $\varepsilon_t$. The changes in the value of $S_t$ will be governed by a probability of transition matrix,

$$\Pi = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & 1-p_{11} \\ 1-p_{22} & p_{22} \end{pmatrix} \tag{53}$$

where,

$$p_{ij} = P\left[S_{t+1} = i \middle| S_t = j\right] \tag{54}$$

This model implies that $A_k$ will be estate dependant while $B_k$ will not change between estates.

$$A_{k+1}(S_{k+1} = j) = -\delta_0 + \frac{1}{2} B_k' \Sigma \Sigma' B_k' + \log \sum_{i=1,2} p_{ij} e^{\left(A_k + B_k' \mu - B_k' \Sigma \lambda_0\right)} \tag{55}$$

$$B_{k+1}' = -\delta_1' + B_k' \Phi - B_k' \Sigma \lambda_1 \tag{56}$$

## 4 RESULTS

To estimate the models proposed in previous section we have used spot nominal interest rate for the Government Yield Curve. This data has been computed from the Yield Curve estimations of the Statistical Department of the Bank of Spain that fits a Nelson and Siegel (1987) model from 1991 to 1996 and a Svensson (1994) since 1996 (Nuñez, 1996)[3]. Interest rates time series considered in our analysis goes from January 1991 to December 1998. The beginning of the sample is determined by the availability of data, while the end is given by the entrance to the European Monetary Union, and nominal interest rates became driven by European determinants. As was showed by Gimeno and Nave (2006) the overlap of the two methodologies in the estimation for the yield curve did not cause any relevant shift.

In the case of the unobserved factor model, given the complexity of the Kalman filter estimation for this amount of parameters, we follow ABW, and restricted the sample to 3-months, 1-year, 3-years and 5-years terms. Moreover, Kalman filter methodology requires to impose two of the interest rates to be measured without error, in order to recover the unobserved latent factors. We have found that, at least in Spain, the results are extremely sensible to the selection of the interest rates observed without error. We choose the 3 months - 5 years interest rates in order to take into account sort and long term interest rates.

Additionally, Kalman filter estimation becomes extremely sensible to the initial values. In the case of real interest rates for economies with some structural changes some robustness problems could arise as stated by Marqués and Manrique (2004). In order to surround such problems some papers like ABW impose a number of ad-hoc restrictions in the parameters. In this paper we have design a real genetic algorithm similar to the one proposed by Gimeno and Nave (2006) that excludes combination of parameters that create non sense estimations ( like negative real interest rate or extremely high values of inflation rates) without imposing ad-hoc restrictions to the parameters.

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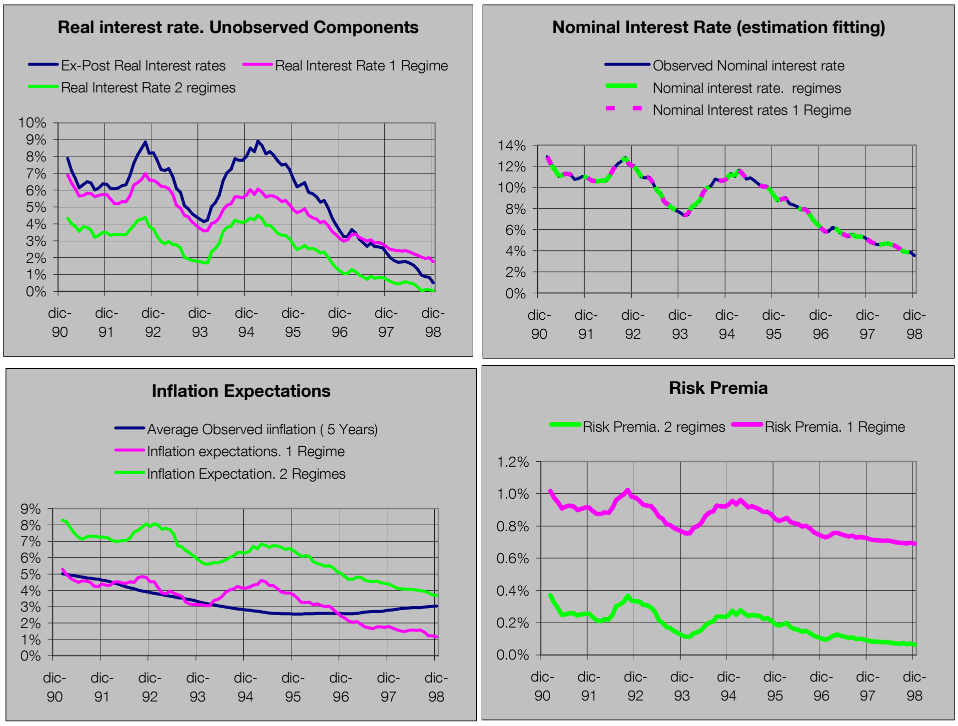
[3] For simplicity in this paper we will label these factors as Nelson&Siegel Factors although part of the sample has been derived by Svensson methodology.

Results for the unobserved component model are presented in the following graph,

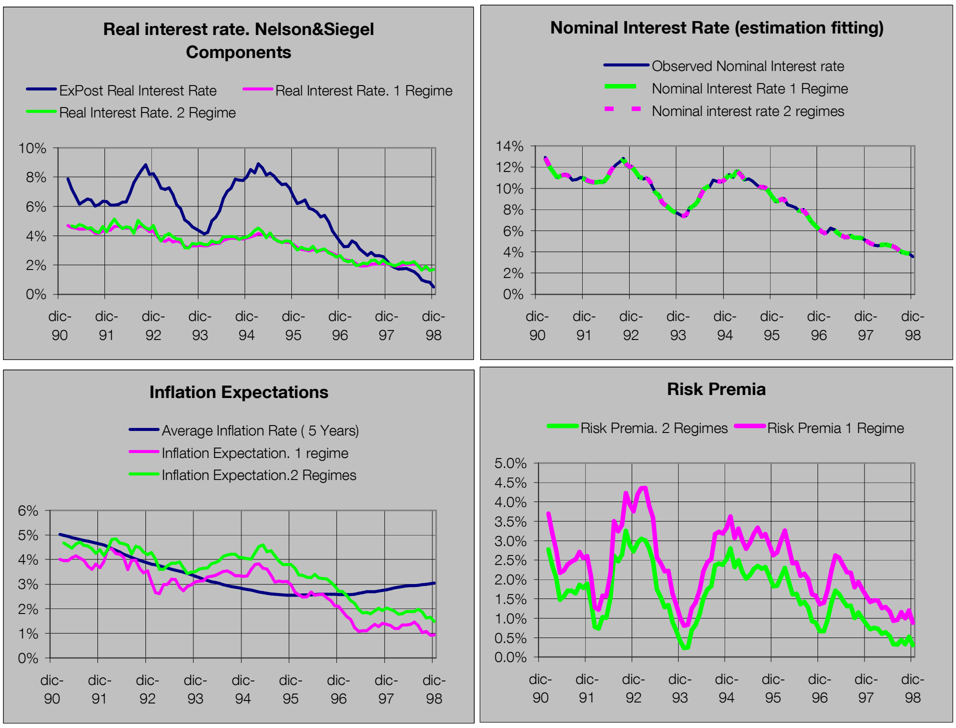
**Unobserved Components Model. 5 Years Interest Rate**



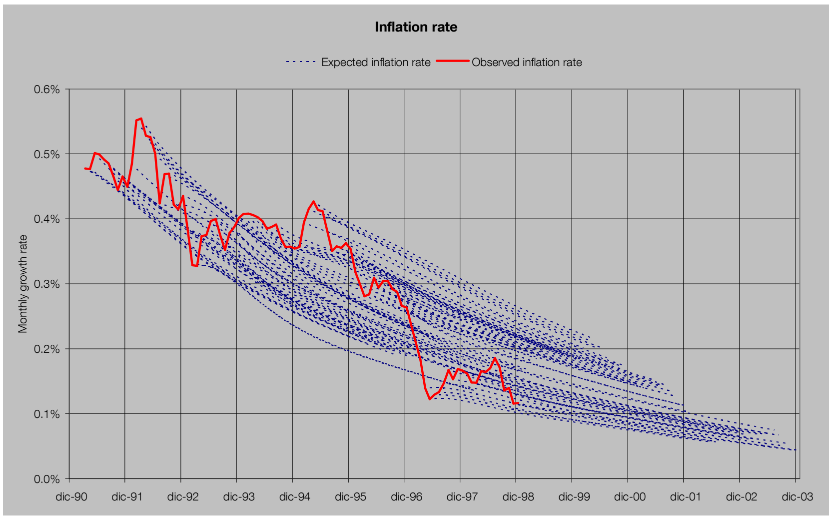
Top-left figure shows that real interest rates has had less volatility and fall in a lower magnitude than can be expected compared with ex-post calculation. This is the consequence of a reduction in the inflation expectation bigger than observed, and a further decline in the risk premium. This result is independent of using or not Markov regime switching, but the level of this three variable seems to be dependent of the specification. The appearance of a second regime implies a shift in the inflation expectation, since the model generates two regimes, one with inflation close to the observed and the other with a high inflation rate. This last regime was not observed in the sample and the results suggest that the markets assign a non constant low probability to its occurrence that end to affect long run variables. Up-right figure shows the goodness of the estimation for the five years nominal spot interest rate. Since this maturity was considered as one of the observed without error is not surprising that the estimation fit properly. However, the magnitude of error terms for other maturities other than 5 years and 3 months could not be neglected and during some periods present important deviations of observed rates.

Results for the observed components model are much more easy to obtain than the previous one since only required a Maximum Likelihood Estimation for Nelson and Siegel parameters and then the whole structures of interest rate could be recover. Therefore the results not only are unaffected by initial parameters assumptions but also suppose an adjustment for all the interest rate along the yield curve. The next graph shows the results with this methodology .It can be shown that the goodness of the estimation is satisfactory and in this framework is possible to reply the term structure. This result goes in line with Diebold and Li (2006) that shows the possibility to estimate the variations of these parameters for United States. Indeed the purpose of this paper was to use this kind of estimation as a forecasting tool. This line goes beyond the purpose of our paper but given the estimation quality is possible to find good forecasting properties for Spanish bonds.

**Nelson & Siegel Components Model. 5 Years Interest Rate**

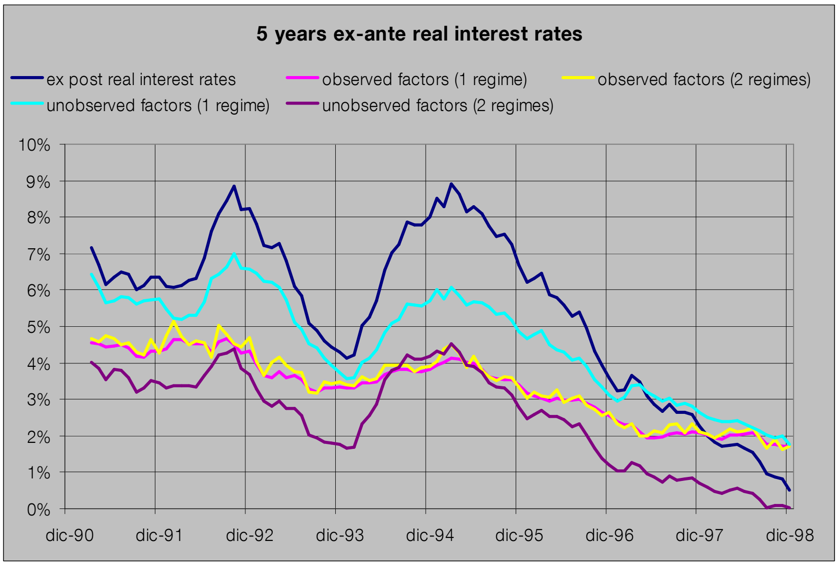


Up-left graph shows that in this case the real interest rate declined only 2 p.p. during the sample period. Moreover this result was obtained independently of the regimes used. The difference in the movement of real ex-post interest rate and risk-free real interest rate could be attributed to the decline in inflation expectation and risk premia. On average the path of the former variable is similar to observed inflation, while the differences are consequence to the variability of the CPI, and the term considered. This could be seen on the next graph where we compared the observed inflation with the projections of this variable derived from equation 4. Moreover both models allow some period where inflation were persistently higher than finally realized, this result goes in line with some concern about the entry of Spain in EMU that could affect not only risk premia but also inflation expectation.



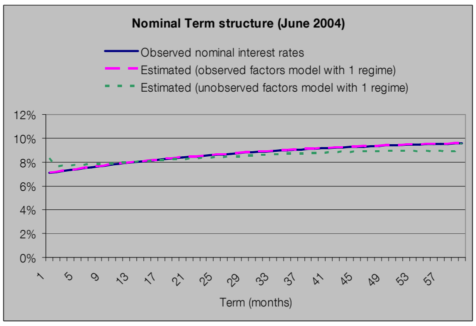
In respect to Risk premia fluctuations they are downward sloping but the level is higher than in previous estimation Indeed an important share of the volatility on nominal rates was reflected in this variable. The two regimes model does not seem to affect the real interest rate estimation. However some changes arise in the decomposition between expected inflation and risk premia. This could be attributed to the fact that the in the two regimes model capture better the risk of been in a non convergence scenario in the level of inflation expectation meanwhile in the one regime model is more difficult to disentangle risk premia and inflation expectation.

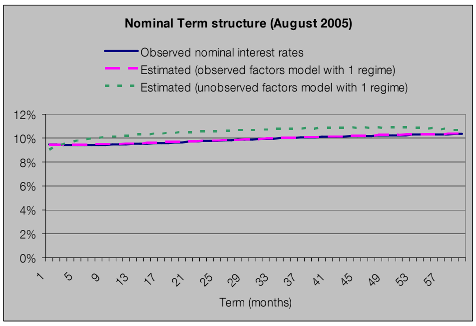
In terms of differences among estimated models, as can be seen in next figure, all of them capture the intuition of Blanco and Restoy (2007) that the evolution of real interest rates in the nominal convergence process supposes a less dramatic fall of interest rates, than ex-post observations would have suggested. However, the level of real interest rates is quite sensible to the model chosen, as showed in next table. Volatility of real interest rate is also higher, when factors are not observed.



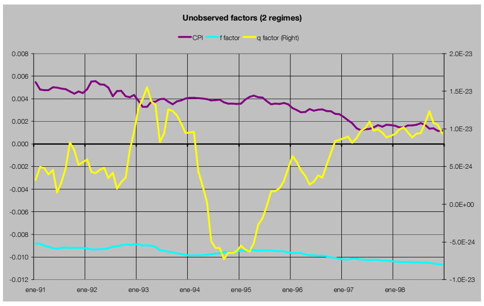
| | Mean interest rate | Std. Deviation | Difference (mar 91-dec 98 |
|---|---|---|---|
| Ex post | 5.43% | 2.28% | 6.67% |
| Ex ante observed factors (1 regime) | 3.32% | 0.92% | 2.85% |
| Ex ante observed factors (2 regimes) | 3.42% | 0.97% | 2.95% |
| Ex ante unobserved factors (1 regime) | 4.55% | 1.40% | 4.66% |
| Ex ante unobserved factors (2 regimes | 2.50% | 1.34% | 3.99% |

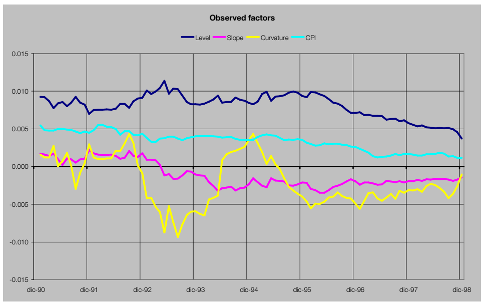
Additionally, as we have previously commented, the goodness of fit of both approaches are quite different, since the observed components take into account the whole structure of nominal interest rates. In next graph, it can be observed for two concrete periods that the model with Nelson and Siegel parameters capture all the term structure, while the unobserved components allows for significant deviations along the yield curve.





The reason behind these fitting problems was already highlighted by Diebold and Li (2006) mentioning that we need three factors in order to reproduce accurately the yield curve. In fact, if we compare the evolution of the factors in both models (next figure), it can be seen that unobserved factors follow similar patterns to the ones of Nelson and Siegel factors. Latent factor *q* matches perfectly the evolution of the curvature parameter, while latent factor *f* seems to be just an imperfect combination of the level and slope parameters.





Therefore, given the dependence of the unobserved model to the initial parameters, the goodness of fit on the yield curve, and the sensibility to imposing regime switching, results are quite more robust and reliable in the case of the model with Nelson and Siegel observed factors. Furthermore, the results for real interest rates obtained with this model are in line with those obtained for other countries and methodologies (Laubach and Williams (2003), Manrique and Marqués (2004), Cuaresma, et al. (2004) among others). Additionally, as Blanco and Restoy (2007) mention, the evolution of Spanish economy would be compatible only with a moderate slowdown in risk-free real interest rates.

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