Warrant pricing with credit risk

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Abstract

The existing bibliography supports the notion that warrant prices depend on the credit risk of the warrant issuer. The purpose of this paper is to value warrants taking into account their issuer's credit risk. We distinguish two types of warrants depending on whether the exercise of the warrant implies dilution of the firm's equity. On the one hand, for pricing warrants with dilution, we extend Ukhov's (2004) model. On the other hand, for valuing warrants without dilution we propose to apply the pricing model for vulnerable options developed in Hull and White (1995). Finally, in order to study the implementation of the expressions we propose, we apply them to price some warrants in the Spanish market.

Journal of Economic Literature classification: G13, G23, G32.

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1 Introduction

In recent years, the negotiation of warrants has undergone rapid growth because of the introduction of the electronic trading system, and their advantages in downward markets. The pricing of warrants is performed by application of the option pricing theory, because of the similarities between these two instruments. Indeed, Black (1989) points out that the Black and Scholes (1973) option pricing model was initially intended to value warrants. However, the theoretical and empirical analysis of warrants has been limited in comparison with the analysis of options. The reason is principally that valuing warrants is more difficult than valuing options. A warrant, like an option, is a right to buy or sell a specified quantity of an asset[1] at an agreed price, on a fixed date or during a specified period of time. In the case of warrants on one's own stocks, and when the exercise of warrants implies the issue of new shares of stock, a dilution of the firm's equity occurs.

Black and Scholes (1973), as an extension of their work, propose to price warrants as stock options. However, other papers study the need to correct the Black and Scholes (1973) expression for dilution. While Sidenius (1996) shows that there is no need for such correction, Schulz and Trautmann (1996), Hauser and Lauterbach (1996) and Ukhov (2004) hold that correction for dilution is required. Examples of papers that correct the Black and Scholes (1973) model are Galai and Schneller (1978), Noreen and Wolfson (1981), revised later in Galai (1989), and Lauterbach and Schultz (1990). However, the expressions used in these articles require the knowledge of the firm's value, which is a function of the warrant price and is not observable. Schulz and Trautmann (1994) and Ukhov (2004) propose a solution for this problem and offer expressions that use observable variables.

In addition to possible dilution, there are other reasons for the difference in prices between warrants and options. Chan and Pinder (2000) state that the credit risk of the issuer of the warrant causes part of this difference. This is because warrants are usually issued by a third party, generally a financial institution, and warrant prices reflect the different levels of credit risk associated with warrant issuers. As an example, in Table 1 the prices of some call warrants on Altadis and Banco Popular traded on the Spanish market are shown. If we compare the price of warrants with same strike price, maturity and ratio, we can observe that the market price is different depending on the issuer of the warrant. In this way, we see that the warrants issued by BBVA have a higher price than those issued by BSCH. Furthermore, Tables 2 and 3 show the rating of the issuers of warrants in Spain and the grading of the rating agencies, respectively. We can check that according to the main rating agencies, BSCH has a higher credit risk than BBVA. So in this case we can say that the higher the credit risk of the issuer, the lower the warrant price.

We can thus consider that the credit risk of the issuer is another factor that influences

[1] We define the *ratio* of a warrant as the number of units of the asset that the holder can receive if he or she exercises the warrant.

the prices of warrants. Chen (2003) introduces the credit risk of the warrant issuer in the pricing of warrants. His article studies the pricing of existing covered warrants in Taiwan, which are a kind of warrants without dilution effect. Chen (2003) considers this type of warrants to be call options issued by a third party and applies the literature about the pricing of options with credit risk.

Traditionally, it has been assumed that options have no default risk. However, many options are sold by firms that have limited assets. For such options, default is often a possibility that must be taken seriously. Johnson and Stulz (1987) were the first to study how options subject to default risk, which they call *vulnerable options*, are priced. Johnson and Stulz (1987) assume that, if the counterparty writing an option is unable to make a promised payment, the holder of a derivative security receives all the assets of the counterparty. However, Hull and White (1995), Klein (1996) and Cao and Wei (2001) find this assumption reasonable only when there are no other claims on the assets of the counterparty that rank equally with the derivative security in the event of a default. Hull and White (1995) extend the Johnson and Stulz model to cover situations where other equal ranking claims can exist. They assume that we know or can estimate the impact of default risk using the prices of bonds that have been issued by the counterparty. Otherwise, Klein (1996) considers that the Hull and White model is only appropriate one when the assets of the counterparty and the asset underlying the option are independent. To solve this limitation, Klein (1996) derives an analytic pricing formula which allows for correlation between the option's underlying asset and the credit risk of the counterparty. This last model is the one that Chen (2003) applies to price the warrants without dilution traded in Taiwan.

There are warrants on different assets on the market. Tables 4 and 5 show the warrants traded on the Spanish market and also their issuers. We can distinguish between warrants on the issuer's own stocks and warrants on other assets. On one hand, in the case of warrants on the issuer's own stocks the exercise can be accompanied by an increase in the number of stocks, which implies a dilution of the firm's equity. Thus, to price one of these warrants, we need to take into account not only the credit risk of the issuer but also the dilution effect. In this paper we propose a model for considering these two factors in the pricing of a warrant with dilution.

On the other hand, in the case of warrants on stocks from other companies or on other assets, the exercise of the warrants does not imply a dilution of the equity. In order to price this kind of warrant, Chen (2003) assumes dependence between the underlying asset and the default risk of the issuer. However, for warrants traded on organized markets or issued by well-diversified institutions, we can suppose independence between the underlying asset and the default risk of the issuer. In this way, the second goal of this paper is to propose a model to value warrants without dilution and with credit risk of the issuer.

The rest of this paper is organized as follows. Section 2 studies the pricing of warrants when the exercise implies a dilution of the equity. Section 3 analyzes the pricing of warrants when there is no dilution. Section 4 applies the proposed pricing formulae to warrants in the Spanish market. Finally, Section 5 concludes the paper.

2 The valuation model for warrants with dilution

In this section we examine the pricing of warrants when dilution of equity occurs at the exercise of the warrant. Although the literature yields different views about the necessity of adjustment for dilution, we can find many papers that include that adjustment. First of all, we analyze the models proposed by these articles. We then develop a pricing model that takes into account both the dilution effect and the credit risk of the issuer.

2.1 Without credit risk

Black and Scholes (1973), as an extension of their work, propose to value warrants as options on the equity of the firm instead of options on the underlying asset. Later, Galai and Schneller (1978), Lauterbach and Schultz (1990) and Crouhy and Galai (1991), consider it necessary to adjust the Black and Scholes formula to take into account the possibility of a dilution as a consequence of the exercise of the warrant. However, the formulae proposed in those works require knowledge of the firm value and of the firm value process variance. When warrants are outstanding, the firm value itself is a function of the warrant price, that is, firm value and firm value variance are then unobservable variables. To solve this problem, Schulz and Trautmann (1994) and Ukhov (2004) propose to value warrants using stock prices and stock-return variance, both observable variables.

Let us now describe the Ukhov (2004) model for valuation of a conventional warrant issued by a firm on its own stock. Suppose that the company has N shares of common stock and M warrants outstanding. Each warrant entitles the owner to receive k shares of stock at time T upon payment of X dollars. These two forms of financing are the only financing the company is issuing. Let V_t be the value of the company's assets, let S_t be the value of the stock and σ_S its volatility. Let $w(V_t,\ T-t)$ denote the value of each warrant at time t. The warrants are exercised only if $kS \geq X$, just as with call options. When all M warrants are exercised, the firm receives MX and issues kM new shares of stock. According to Galai and Schneller (1978) and Ingersoll (1994), if the Black and Scholes assumptions are satisfied, the value of a European call warrant is:

$$w(V_t,\ T-t,\ X,\ \sigma,\ r,\ k,\ N,\ M) = \frac{1}{N+kM}\left[kV_tN(d_1) - e^{-r(T-t)}NXN(d_2)\right] \tag{1}$$

with:

$$d_1 = \frac{\ln(kV_t/NX) + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{2}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{3}$$

where σ is the standard deviation of V_t.

In order to know the value of σ and following Ingersoll (1994), Ukhov (2004) relates the value of σ to the volatility of the underlying σ_S. Define Ω_S, with $\Omega_S \equiv \Delta_S V_t/S_t$, as the elasticity of the stock price with respect to the firm value, where $\Delta_S = \frac{\partial S_t}{\partial V_t}$. This way, stock volatility is related to firm volatility via:

$$\sigma_S = \Omega_S \, \sigma = (\Delta_S V_t / S_t)\sigma \Leftrightarrow \sigma = \frac{\sigma_S}{\Delta_S V_t / S_t} \tag{4}$$

In this model, the firm has N shares of stock and M warrants, that is, $V_t = N\, S_t + M\, w_t$. Hence, when the firm value changes by 1 dollar, the warrant's price changes by Δ_w and the stock price changes by Δ_S. Then,

$$M\Delta_w + N\Delta_S = \Delta_V = 1 \tag{5}$$

To obtain the expression for Δ_S we need first Δ_w. If equation (1) is used, we have:

$$\Delta_w = \frac{\partial w(V_t;\, \cdot)}{\partial V_t} = \frac{k}{N + kM} N(d_1) \tag{6}$$

And by substituting in (5) we obtain the expression for Δ_S:

$$\Delta_S = \frac{1 - M\Delta_w}{N} = \frac{N + kM - kMN(d_1)}{N(N + kM)} \tag{7}$$

Finally, if we substitute (7) in (4), we obtain the relationship between σ and σ_S.

Once related σ to σ_S, Ukhov (2004) proposes this algorithm for computing the warrant price from the observed variables S_t and σ_S:

1. Solve (numerically) the following system of nonlinear equations for (V_t^*, σ^*):

$$\begin{cases} N\, S_t = V_t - M\, w(V_t,\, T - t;\, X,\, \sigma,\, r,\, k,\, N,\, M) \\ \sigma_S = \frac{V_t}{S_t} \Delta_S \sigma \end{cases} \tag{8}$$

with:

$$\Delta_S = \frac{N + kM - kMN(d_1)}{N(N + kM)} \tag{9}$$

2. The warrant price is obtained as:

$$w = \frac{V_t^* - N\, S_t}{M} \tag{10}$$

Ukhov (2004) shows that the system (8) has a solution $(\sigma^*, V_t^*) \in (0, +\infty) \times (0, +\infty)$.

As we can see, Ukhov (2004) develops a pricing model for warrants on the issuer's own stocks that takes into account the dilution of equity. Moreover, to solve the limitations found in other works, Ukhov (2004) uses observable variables. However, he does not consider that the warrant issuer may also be financed by debt. That is, Ukhov (2004) does not take into account the credit risk of the issuer.

2.2 With credit risk

In what follows we shall extend Ukhov's model for pricing European call warrants on the firm's own stocks when the issuer is financed by debt.

Consider a firm with M warrants, N shares of stock and a zero-coupon bond with face value F and maturity at T. Let us suppose that the maturity date is the same for warrants as for the bond. The owner of the warrant has the right to pay X at T and receive k shares of stock with individual value $\frac{1}{N+kM}(E_T + MX)$, where E_T is the value of equity at T, just after the exercise of warrants. In this way, we have that the value of the warrant at T is:

$$w_T = Max(0,\ k\lambda(E_T + MX) - X) \tag{11}$$

where $\lambda = \frac{1}{N+kM}$. Furthermore, we know that the value of equity at T is $E_T = Max(V_T - F, 0)$, because if the value of the company at T is larger than the face value of debt, F, debtholders get F while shareholders get $V_T - F$, and in case of default, the debtholders receive what is left of the company, V_T, while the shareholders get 0. Thus, we can write (11) in this way:

$$w_T = Max\big(0,\ Max(k\lambda(V_T - F + MX) - X, -\lambda NX)\big) \tag{12}$$

Since the values of λ, k, N and X are bigger or equal to zero, the expression for w_T can be written as follows:

$$w_T = \lambda Max(0, kV_T - kF - NX) \tag{13}$$

Thus, the value of the warrant at t, w_t, must satisfy:

$$w_t = \lambda c(kV_t, kF + NX, \sigma) \tag{14}$$

where $c(\cdot)$ represents the value of a European call option on kV_t, with strike price $kF + NX$, that is:

$$w(V_t,\ \sigma,\ X) = \frac{1}{N+kM}\left[kV_tN(d_1) - e^{-r(T-t)}(kF + NX)N(d_2)\right] \tag{15}$$

with:

$$d_1 = \frac{\ln\left(\frac{kV_t}{kF+NX}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{16}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{17}$$

This expression requires as an input the firm value, V_t, and its volatility, σ, that are unobservable variables. On the basis of Ukhov (2004), we search for a relationship between V_t and σ with S_t and σ_S, which are observable variables. As we have seen before, we can establish a relationship by this expression:

$$\sigma_S = \frac{V_t}{S_t}\Delta_S\sigma \tag{18}$$

To compute Δ_S when there exists debt we see that now $V_t = NS_t + Mw_t + D_t$, so we have that:

$$\Delta_V = 1 = N\Delta_S + M\Delta_w + \Delta_D \tag{19}$$

Using (15) we obtain:

$$\Delta_w = \frac{\partial w_t}{\partial V_t} = k\lambda N(d_1) \tag{20}$$

where:

$$d_1 = \frac{\ln\left(\frac{kV_t}{kF+NX}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{21}$$

On the other hand, to obtain the expresssion for Δ_D first we must determine the expression for D_t. It is known that at maturity debtholders receive this flow:

$$D_T = min(F,\ V_T) = F - Max(0,\ F - V_T) \tag{22}$$

then, D_t is given by

$$D_t = Fe^{-r(T-t)} - p(V_t, \sigma, F) \tag{23}$$

where $p(V_t, \sigma, F)$ is the value of a European put option on the value of the firm with strike price F. Thus, Δ_D is given by this expression:

$$\Delta_D = \frac{\partial D_t}{\partial V_t} = 1 - N(f_1) \tag{24}$$

where:

$$f_1 = \frac{\ln\left(\frac{V_t}{F}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{25}$$

Once we know the expressions for Δ_w and Δ_D and substituting in (19), we have V_t related to σ, S_t and σ_S when the firm is financed by equity, warrants and debt.

Furthermore, we can consider that the stockholders and the owners of the warrants have a European call option on the value of the firm, with exercise price equal to the face value of the debt, and with maturity at T, that is, $NS_t + Mw_t = c(V_t,\ \sigma,\ F)$. Moreover, using the put-call parity we can check that $V_t = NS_t + Mw_t + D_t$ is satisfied.

Once we have established these conditions, we obtain the following result:

Result 1 *Suppose a company with value V_t, financed by N shares of stock, M European call warrants and a zero-coupon bond with face value F and maturity at T. Each warrant entitles the owner to receive k shares of stock upon the payment of X at time T. Let S_t be the price of a stock and let σ_S be its standard deviation. The value of a European call warrant at time t is given by the following algorithm:*

1. *Solve (numerically) the following system of nonlinear equations for* (V_t^*, σ^*)*:*

$$\begin{cases} N\, S_t = V_t N(f_1) - e^{-r(T-t)} F N(f_2) - M\lambda \left[kV_t N(d_1) - e^{-r(T-t)} (kF + NX) N(d_2) \right] \\ \sigma_S = \frac{V_t}{S_t} \Delta_S \sigma \end{cases} \tag{26}$$

with:

$$\Delta_S = \frac{N(f_1) - \frac{kM}{N+kM} N(d_1)}{N} \tag{27}$$

and where:

$$f_1 = \frac{\ln\left(\frac{V_t}{F}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{28}$$

$$f_2 = f_1 - \sigma\sqrt{T-t} \tag{29}$$

$$d_1 = \frac{\ln\left(\frac{kV_t}{kF+NX}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{30}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{31}$$

2. *The warrant price at t is obtained as*

$$w(V_t^*,\, \sigma^*,\, X) = \frac{1}{N+kM} \left[kV_t^* N(d_1) - e^{-r(T-t)} (kF + NX) N(d_2) \right] \tag{32}$$

Figure 1 shows the pricing of a call warrant taking into account the effect of dilution and the credit risk of the issuer. We see that the bigger the stock price, the higher the warrant value. We must note that the firm value that satisfies the proposed system of equations, V_t^*, increases with S_t. That is, while for $S_t = 2$ we obtain $V_t^* = 21{,}850$, for $S_t = 4$ we have $V_t^* = 41{,}974$. We must be careful in the study of the effect of debt on the price of the warrant, because V_t^* also changes if we modify the value of F. After fixing some conditions, we provide an example of the influence of F on the price of the warrant, that can be seen in the Appendix.

Figure 2 considers the case of a firm without debt. We study the pricing of warrants and also the effect of dilution on the price of the warrant. In the upper graphic, we compare the value of a warrant when the firm is financed with 10,000 shares of stock and only one warrant, and when the firm has 10,000 shares of stock and 1,000 warrants. In the first case the effect of dilution is minimal, that is, $\frac{M}{N} \to 0$. The proposed pricing formula coincides with the model of Black and Scholes (1973). In the second case, the proposed formula is the same as the formula proposed by Ukhov (2004). Our model nests the models of Black and Scholes (1973) and Ukhov (2004). Moreover, if we compare the two cases studied in the first graph, we observe that dilution hardly affects the price of the warrant. This result is consistent with Sidenius (1996), who finds that there is no need to correct the Black and Scholes (1973) formula for dilution. In the lower graphic we analyze the effect of more pronounced dilution on the warrant price. We compare the value of a warrant when the firm has only one warrant and when the firm has 7,000 warrants outstanding. The effect of dilution is now noticeable. Moreover, if we compare the two cases, we can observe that the greater the dilution, the lower the warrant price.

3 The valuation model for warrants without dilution

Besides warrants on the issuer's own stocks, there exist warrants on stocks from other companies, and warrants on other assets, such as currencies, commodities, indexes, interest rates or exchange rates. In the case of warrants on the issuer's own stocks, at exercise the firm can deliver existing stocks, without issuing new shares of stock. Thus the exercise of all these warrants does not provoke a dilution of the equity. In this section we want to develop a model for pricing warrants without dilution and with credit risk of the issuer.

Chen (2003) applies Klein's (1996) model for pricing warrants without dilution and with credit risk of the issuer. Klein (1996) develops a model for valuing vulnerable options that assumes that the underlying asset is correlated with the credit risk of the issuer of the option. According to Chen (2003), even if the assets of the counterparty have not deteriorated during the option's lifetime, credit risk may still exist if the assets have not grown sufficiently to make the promised payment to the in-the-money option holders because of the rise in the underlying stock price. The correlation between the value of the underlying security and the asset value of the counterparty is thus also an important factor to determine the counterparty's default risk and should be incorporated into the vulnerable option pricing model. In this way, Klein (1996) extends the models of Hull and White (1995) and Jarrow and Turnbull (1995) to allow for correlation between the credit risk of the counterparty and the asset underlying the option.

However, the dependence assumption is inappropriate when the derivative writer is a large well-diversified financial institution, as noted by Hull and White (1995), or when the derivative position is insignificant on an individual basis as compared to the total assets of the counterparty, as Klein (1996) says.

In this section we intend to price warrants issued by well-diversified financial institutions, so we can assume independence between the default risk of the issuer and the underlying asset. For valuing these warrants we propose to use the vulnerable option pricing model developed in Hull and White (1995).

3.1 Hull and White (1995) model

In their model for pricing vulnerable options, Hull and White (1995) modify the assumption made by Johnson and Stulz (1987) that the holder of a derivative security receives all the assets of the counterparty in the event of a default. Thus, in Hull and White (1995), the holder of an option is assumed to recover a proportion of its no-default value in the event of default by the counterparty. Both the probability of default and the size of the proportional recovery are random.

The Hull and White (1995) model requires knowledge of the default boundary, the proportion of no-default value received in case of default and the probability of the occurrence of default[2]. However, in practice these data are unlikely to be available. To solve this difficulty, Hull and White (1995) consider a special case of the model where the adjustments for credit risk depend only on the prices of bonds.

The special case supposes independence between the variables that determine the value of the default-free option (θ variables) and the variables determining the occurrence of defaults and the payoff received in the event of default (ϕ variables). In this way Hull and White (1995) obtain this expression:

$$f = \frac{B}{B^*} f^* \tag{33}$$

where f is the current value of the vulnerable option, f^* is the current value of the option assuming no defaults, B is the current value of a vulnerable zero-coupon bond issued by the option writer that pays off 1 dollar at maturity and ranks equally with the option in the event of a default[3], and B^* is the current value of a similar default-free zero-coupon bond. The intuition behind equation (33) is that the proportional loss on the bond and the option because of the chance of default is the same. Since θ and ϕ variables are independent, the expected no-default values of the bond and the option are independent of the path followed by ϕ variables. Defining y and r as the yields on B and B^* respectively, where r is the risk-free interest rate, (33) is reduced to:

$$f = e^{-(y-r)(T-t)} f^* \tag{34}$$

This expression suggests that the discount rates used when a vulnerable option is valued should be higher than those used when a similar default-free option is valued by an amount $y - r$.

The independence assumption may in practice not be too unreasonable for many of the over-the-counter options written by large financial institutions. One reason for this is that any particular option is usually only a very small part of the portfolio of the financial institution. Another reason is that the variables underlying the options traded over the

[2] See Hull and White (1995) for more details.

[3] We define two securities as *ranking equally in the event of a default* if both pay off the same proportion of their no-default values when a default occurs.

counter by financial institutions are typically interest rates, exchange rates and commodity prices. Most financial institutions try to ensure that they are well hedged against the impact of these market variables. Expression (34) leads to the following modifications to the Black and Scholes (1973) formula:

$$c(S_t, K, T) = e^{-(y-r)(T-t)}\left[S_t N(d_1) - Ke^{-r(T-t)}N(d_2)\right] =$$

$$= e^{-(y-r)(T-t)}S_t N(d_1) - Ke^{-y(T-t)}N(d_2) \tag{35}$$

with:

$$d_1 = \frac{\ln\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{36}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{37}$$

and where S_t is the stock price at t, K is the strike price, σ is the volatility, $N(\cdot)$ is the cummulative normal distribution function, y is the yield of the risky debt of the issuer and r is the yield of the default-free debt of the issuer, that is, the risk-free interest rate.

3.2 Valuation of warrants without dilution using the Hull and White (1995) model

As we have just seen, the model of Hull and White (1995) for pricing vulnerable options introduces the credit risk of the issuer through the difference between the yield of debt and the risk-free interest rate. Therefore, to apply Hull and White (1995) we must know this spread, given by $y - r$. In order to compute y we need to know the yield to maturity of the zero-coupon bonds issued by the issuer when there exists the possibility of default. We also must notice that a warrant gives the holder the right to receive k shares of stock upon the payment of X, that is, we need to introduce the ratio of the warrant, k, in expressions (35) - (37).

Consider a European call warrant on an asset with price S_t and volatility of its return σ_S. Each warrant entitles the owner to receive k units of the asset upon the payment of X dollars and with maturity at T. Let y be the yield to maturity of a zero-coupon bond issued by the issuer of the warrant and let r be the risk-free rate. The value of this warrant at t is given by the following expression:

$$w(S_t,\ T-t;\ X,\ \sigma_S,\ y,\ r,\ k) = e^{-(y-r)(T-t)}\left[k\ S_t N(d_1) - Xe^{-r(T-t)}N(d_2)\right] =$$

$$= \left[e^{-(y-r)(T-t)}\ k\ S_t N(d_1) - Xe^{-y(T-t)}N(d_2)\right] \tag{38}$$

with:

$$d_1 = \frac{\ln\left(\frac{k\ S_t}{X}\right) + \left(r + \frac{\sigma_S^2}{2}\right)(T-t)}{\sigma_S\sqrt{T-t}} \tag{39}$$

$$d_2 = d_1 - \sigma_S \sqrt{T-t} \tag{40}$$

To obtain the values of parameteres appearing in expressions (38) - (40) we can use the information on the issuing of debt by the warrant issuer. Thus, contrary to Klein's model (1996), which requires knowledge of costs of default and default thresholds, the application of Hull and White (1995) to the pricing of warrants only requires knowledge of variables that can easily be observed.

Figure 3 shows the pricing of a warrant as a function of the price of the underlying asset. We observe that the bigger the price of the asset, the smaller the value of the warrant. Figure 4 illustrates the effect of the issuer's level of credit risk on the price of the warrant. The higher the difference between the yield of debt with and without risk, the smaller the price obtained for the warrant. This result is consistent with Klein (1996) and Jarrow and Turnbull (1995), who obtain a reduction in option prices due to the credit risk of the counterparty. The reason for that reduction is that in case of default, the derivative holder receives only a proportion of the amount he or she would receive if default does not occur. In this figure we also show as a special case the valuation of the warrant when there is no credit risk, that is, $y = r$. In this case, the pricing formula we propose is equal to the Black and Scholes (1973) formula.

The way of introducing credit risk when there is dilution is thus different from the way when there is no dilution. On one hand, if the exercise of a warrant is accompanied by the issue of new shares and there exists a dilution of the equity of the firm, we extend the Ukhov model (2004). On the other hand, when there is no dilution, we use the credit spread of the issuer to take into account the credit risk. Thus, we propose two alternative ways of considering the credit risk of the issuer. Furthermore, we show that these alternative expressions coincide with the Black and Scholes (1973) model when there is no credit risk or dilution.

4 Pricing warrants in the Spanish market

In this section we study the implementation of the formulae we have proposed for reflecting the effect of credit risk on warrant prices. These expressions value only European call warrants. However, for warrants on non-dividend paying stocks the price of an American call warrant is the same as the price of a European call warrant. In the case of warrants on dividend paying stocks, we can use our expressions as an approximation to the value of the American warrant.

Next, to show the implementation of the proposed formulae, we value some warrants traded in the Spanish market. The Spanish warrants market has undergone considerable growth in recent years. In 1996 the number of warrants traded in Spain was 5.11 million with a value of 16.31 million euros, while in 2005 the negotiation was 4,020 million warrants with a value of 2,142 million euros[4].

[4]Source: Review of the Madrid Stock Exchange, issues 122 and 150.

In spite of the developments in the Spanish warrants market, there is almost no empirical research. Morillo (2003) reviews the main models of warrant pricing and applies Black and Scholes (1973) to price some warrants traded in Spain. However, he does not consider the effect of dilution on the warrant prices, or the effect of credit risk. Another work is Abad and Nieto (2006), who study the differences between the prices of options and warrants in the Spanish market. They obtain that for options and warrants with the same characteristics, warrants are overpriced with respect to options. They also compare prices of equivalent warrants but issued by different entities, and they obtain significant divergences in prices. However, they do not study the effect of the issuer's credit risk on these differences.

To extend the analysis of Morillo (2003) and Abad and Nieto (2006), we value warrants traded in Spain taking into account the credit risk of the issuer. First, we price warrants on the issuer's own stocks by using our extension of Ukhov's (2004) model. As mentioned before, in this extension we introduce the credit risk through the face value of the issuer's debt. Thus, the bigger the face value of the debt, the higher the credit risk and the smaller the price of warrants. Second, we apply the model of Hull and White (1995) to the pricing of warrants without dilution. In this case, we introduce the credit risk through the credit spread of the issuer of the warrant.

4.1 Implementation of the extension of Ukhov's (2004) model

Consider the warrants on own equity issued by Sogecable. These warrants were issued in 2003 with maturity in 2012. From Table 4 we see that these warrants are American-style warrants. Since Sogecable has never paid dividends, we can suppose no payment of dividends during the lifetime of the warrant. We can therefore use the extension proposed to price European warrants for the pricing of these American warrants.

According to the information provided at the issue of the warrants, Sogecable points out that the exercise can be carried out by delivering stocks to the warrant holders or by paying cash. Let us suppose that, at exercise, Sogecable issues new equity to deliver new stocks to the warrant holders. That is, we are assuming that the exercise implies dilution of the equity of the firm.

In order to price the Sogecable warrants, we will use the extension of Ukhov (2004) that we have developed. The pricing date is December 16, 2005. On this date, there is only a call warrant of Sogecable on its own stocks, with strike price equal to 25.76 euros. As the risk-free rate we take the Treasury bonds rate with maturity in five years, because it is the nearest period of time to the lifetime of a warrant. Thus, $r = 0.0303$. To estimate the face value of the debt of Sogecable, F, we capitalize until maturity the value of bank debt of Sogecable on September 30, 2005, $1,010.1$ million euros, plus the loan of $1,200$ million euros received on July 15, 2005[5]. The interest rate for this capitalization is the yield to maturity of the loan received by Sogecable, that is, 0.0314. We must notice that

[5] Source: Consolidated Results of Sogecable on September 30, 2005.

this rate is greater than the risk-free rate. This fact shows the credit risk inherent to the debt issued by Sogecable. Moreover, at the pricing date, the number of outstanding stocks and warrants of Sogecable is $N = 133,564,631$ and $M = 1,260,631$, respectively.

To approximate the value of σ_S we follow the usual practice of using the implied volatility instead of the historical volatility. The implied volatility is a summary of expectations of the market participants about future volatility. To minimize the pricing error, we use the implied volatility of the day before. We use closing prices for warrants as well as for the underlying stock. Additionally, to compare our extension with other models that do not take into account the effect of credit risk, we need to know the value of parameter σ_S of the Ukhov (2004) and Black and Scholes (1973) formulae. With this aim in mind, we also use the implied volatility of the stock price.

Table 6 shows the implied volatility for each model. The three first columns indicate the characteristics of the warrants to be priced. The next three columns show the value of the implied volatility of the stock for the extension we develop, Ukhov's (2004) model and Black and Scholes (1973) model, respectively. In the case of both our extension and Ukhov's (2004) model, we must remark that to obtain the implied volatility we need to know the value of the firm at t, V_t. To estimate V_t, we use the value of the firm according to the consolidated results of Sogecable on September 30, 2005.

Once we obtain all the parameters appearing in expressions (26) - (32), we price the call warrant of Sogecable with strike 25.76 euros. Table 7 offers the results. The first three columns show the characteristics of the warrant. The next three columns provide the result from the pricing of the warrant with our extension, Ukhov (2004) model and Black and Scholes (1973) model, respectively. Finally, in the last column the market price appears. We also indicate the value for V_t^* that satisfies the system of equations that appears in our extension as well as in Ukhov's (2004) model.

We observe that the prices we obtain with our extension fit the market prices better than the prices given by Ukhov's (2004) model. We must note that Ukhov's model offers a higher price than our extension. This is partially explained because V_t^* is bigger for Ukhov's (2004) model than for our extension. Both our extension and Ukhov's (2004) model overprice the warrant, compared to the market price. We must remember that the implied volatility is greater for these models than for the Black and Scholes (1973) model. This is the reason why with the Black and Scholes (1973) model we obtain a better adjustment to the market price. We therefore observe that the three models overprice the warrant.

4.2 Application of the model of Hull and White (1995)

In this section we apply the model proposed by Hull and White (1995) to the pricing of some of the warrants without dilution traded in the Spanish market. Concretely, we price warrants on shares of stock and some warrants on index IBEX-35. The pricing date is December 16, 2005.

First, we value the warrants on stocks. We price warrants from Table 1 and some warrants on Endesa, Iberdrola and Repsol. We price warrants issued by BBVA, Banesto and BSCH, with maturity in March or June 2006. Before applying the Hull and White (1995) formula, we require the value of $y-r$ for each warrant issuer. As the risk-free interest rate, r, we use the rate of Treasury Bonds with maturity in three or six months, depending on the maturity of the warrant. Thus, $r = 0.0207$ for warrants with maturity in March 2006 and $r = 0.0217$ for warrants with maturity in June 2006. To know the yield of risky debt, y, we look at the issues of bonds performed by the issuers of warrants. We use debt issues with similar maturity and that are alive at the date of pricing. Thus, we consider two-year bonds issued by Banesto in October 2004, by BBVA in September 2005 and by BSCH in February 2004. Furthermore, these bonds cover the lifetime of each warrant. They thus represent the credit risk of the issuers during the lifetime of the warrants. The yields of these bonds depend on the Euribor interest rate with maturity in three months. Once we know that rate on the date of issue of each bond, we obtain that $y = 0.024898$ for Banesto, $y = 0.0217$ for BBVA and $y = 0.024273$ for BSCH. On the first hand, we can observe that Banesto, with a higher credit risk according to the main rating agencies (see Table 2), has a bigger credit spread. On the other hand, BBVA, with a smaller credit risk according to the rating agencies, has a smaller credit spread. This result is consistent with that of Longstaff, Mithal and Neis (2005), who find that the majority of the corporate spread is due to default risk. Micu, Remolona and Wooldridge (2004) point out that because rating agencies have privileged access to information about borrowers, investors perceive that rating agencies enjoy an informational advantage. Thus, according to Micu, Remolona and Wooldridge (2004), rating events should have an immediate impact on credit spreads.

Once we know $y-r$, we apply the Hull and White (1995) model to the pricing of these warrants. Since they are warrants on shares of stock, their value can be affected by the payments of dividends. Following Sterk (1983), Kremer and Roenfeldt (1992) support the use of the Merton adjustment for dividends. However, Schwartz (1977) and Leonard and Solt (1990) obtain that the Black and Scholes (1973) model fits market prices better if it is not adjusted for dividends. On the basis of these studies, we take as the price in the formulae we propose the price of the underlying stock minus the current value of its dividend payments during the lifetime of the warrant. We must thus estimate the date and amount of the payments of dividends during the lifetime of the warrant. Harvey and Whaley (1992) show that procedures that assume payments of dividends at a constant rate can produce large pricing errors, since dividends have a seasonal pattern. They suggest to construct a dividend series based on historical payments. Table 8 offers the dates of payment of dividends as well as the amount paid by each stock in the period between January 1, 2000 and December 15, 2005. We observe that Banco Popular pays quarterly dividends, Endesa, Iberdrola and Repsol half-yearly, while Altadis pays dividends in the first two quarters of the year. Using this information, we estimate the dates and amounts paid during the time to maturity of each warrant. In the case of Altadis, we suppose that next payment is on March 22, 2006, that is, this company will not pay any dividend during the time to maturity of the warrant. For the rest of the stocks we study, we assume that dates and payments are the same as in 2005, as shown in Table 9.

Before applying the Hull and White (1995) model, we need to know the value of σ_S. As in the case of the warrants on Sogecable, we use the implied volatility of the day before the pricing date. We use the closing price for both the underlying asset and the warrant. In order to compare with the Black and Scholes (1973) model, we need to know the value of σ_S for this model. We use also the implied volatility. In Table 10 we show the implied volatilities for the models proposed by Hull and White (1995) and Black and Scholes (1973). The first column shows the asset underlying the warrant. The next four columns indicate the characteristics of the warrant. Finally, in the last two columns we show the implied volatility for both models.

Once we know r, y, S_t and σ_S, we value the warrants. The warrants are American warrants, while the proposed model is for pricing European warrants. However, when the underlying asset pays no dividends during the lifetime of the warrant we can use the Hull and White (1995) model for pricing the American warrants. If there is payment of dividends, we can use the price given by Hull and White (1995) as an approximation of the price of the American warrants. Tables 11 - 15 provide the results from the application of the model we propose to price warrants with credit risk and without dilution, that is, the model of Hull and White (1995). The results of the Black and Scholes model (1973), without considering the credit risk of the issuer, are also provided. Finally, the market prices of the warrants are shown. We divide the results according to the moneyness of the warrant, that is the ratio between the strike price and the market price of the underlying asset. We use the classification of the degree of moneyness proposed by Peña, Rubio and Serna (2001): *deep in the money* (0.90-0.97), *in the money* (0.97-0.99), *at the money* (0.99-1.01), *out of the money* (1.01-1.03) and *deep out of the money* (1.03-1.08). We use the closing prices for both the warrant and the underlying asset.

By using the Hull and White (1995) model, we get warrant prices that are similar to the market prices. See for example the prices that we obtain for the warrants issued by BSCH on Banco Popular, Endesa and Iberdrola, and the warrant issued by BBVA on Iberdrola. Moreover, in the case of the warrants issued by BSCH on Endesa and Iberdrola, and the warrant issued by BBVA on Repsol, we obtain a better adjustment to the market price than with Black and Scholes (1973) model. For other warrants, such as the ones issued by BBVA on Endesa and Iberdrola and the warrant issued by BSCH on Banco Popular, the price given by the Hull and White (1995) model is the same as the price given by Black and Scholes (1973).

Next, we turn to the pricing of the warrants on the index IBEX-35 issued by BBVA, Banesto and BSCH, with maturity in March and June 2006. We also use as the risk-free rate the rate of Treasury Bonds, that is, $r = 0.0207$ for the warrants with maturity in March and $r = 0.0217$ for the warrants with maturity in June. Moreover, we have $y = 0.024898$ for Banesto, $y = 0.0217$ for BBVA, and $y = 0.024273$ for BSCH.

To approximate the volatility we use the implied volatility of the day before. Table 16 shows the implied volatilities for warrants on IBEX-35. The first four columns indicate the characteristics of the warrants. The fifth column shows the implied volatility for the model of Hull and White (1995), that is, for the model that considers the credit risk of the issuer. Finally, in the last column we offer the implied volatility for the model of Black and Scholes (1973), that does not take the effect of the credit risk of the issuer into consideration.

Table 17 provides the results of the pricing. We observe that in the case of warrants that are at the money we get a better fit with the market price than in the case of warrants that are deep in the money. We notice that for the warrants issued by BBVA we obtain the same price with the Hull and White (1995) and the Black and Scholes (1973) models. The reason is that the yield to maturity of BBVA's debt is almost the same as the risk-free interest rate. Finally, we must remark that both models, Hull and White (1995) and Black and Scholes (1973), give a lower price than the market price for the warrants on the IBEX-35 index.

5 Conclusions

This paper proposes two alternative methods for pricing European call warrants taking into account the credit risk of the issuer. The proposed formulae distinguish between warrants with dilution and warrants without dilution.

For warrants on the issuer's own stocks and when their exercise implies the issue of new shares of stock, we have extended the Ukhov (2004) model, which prices warrants using stock prices and stock return variance. Ukhov (2004) solves the problem found in the classical warrant pricing, which requires the knowledge of unobservable variables. The extension we develop consists of the introduction of debt in the financing of the firm. In this way we take into account the credit risk of the issuer in the pricing of warrants.

In the case of warrants on the issuer's own stocks but when the exercise does not imply an issue of equity, or in the case of warrants on indexes, commodities, interest rates, currencies, exchange rates or stocks from other companies, there is no dilution as a consequence of the exercise of the warrant. For valuing these warrants and to reflect the credit risk of the issuer, we have proposed to apply the model for pricing vulnerable options developed by Hull and White (1995). They incorporate credit risk using data on bonds issued by the counterparty of an option. The application we propose consists of an alternative to the model used by Chen (2003), which requires the knowledge of costs of default and default thresholds.

Moreover, we have analyzed the influence of credit risk and dilution on the prices of warrants. On the one hand, the greater the dilution the smaller the price of the warrant, because the loss of shareholder value is greater. On the other hand, the higher the credit risk, the lower the value of the warrant. This result is consistent with evidence found by Klein (1996) and Jarrow and Turnbull (1995). Furthermore, we show that the expressions we propose coincide with the Black and Scholes (1973) formula when there is no credit risk on the part of the issuer, or dilution.

Finally, in order to study the implementation of the formulae we propose, we have valued some warrants traded in the Spanish market. First, we have applied the extension of Ukhov (2004) to the pricing of warrants issued by Sogecable on its own stocks. Second, we have applied Hull and White's (1995) model to the pricing of warrants on index IBEX-35 and on stocks from other companies. We have shown that market prices usually reflect the credit risk of the issuer, obtaining a lower price for warrants with higher credit risk of the issuer.

Appendix: Effect of debt leverage on the price of a warrant with dilution

Consider a firm with value V_t, that has N shares of common stock, M European call warrants with maturity at T, and a zero-coupon bond with face value F and maturity at T. Each warrant entitles the owner to receive k shares of stock upon payment of X dollars. Let us suppose that exercise implies the dilution of the equity of the firm.

Now we want to study the effect of a change in the amount of debt issued by the company on the price of one of their warrants. With this goal in mind, we consider a special situation with the following assumptions:

1. Modigliani and Miller (1958) Theorem holds, that is, the firm value is independent of its capital structure.

2. When the firm issues new debt, the firm repurchases stock, satisfying this equation:

$$N_r = \frac{D' - D}{S_t} \tag{1}$$

   where D' is the current value of new debt with face value F', D is the current value of initial debt with face value F, N_r is the number of stocks that the firm buys and S_t is the price of each share of stock at time t.

3. When there is a change in the number of stocks of the firm, the ratio of the warrants changes in this way:

$$k' = k\frac{N - N_r}{N} \tag{2}$$

   where k' is the new ratio.

4. The stock-return variance remains constant.

Thus, since Modigliani-Miller Theorem holds, the firm value, V_t, does not change. We are now ready to state the following algorithm to obtain the price of the warrant:

1. Solve (numerically) the following system of nonlinear equation for (S_t^*, σ^*):

$$\begin{cases} (N - N_r)\, S_t = \\ = V_t N(f_1) - e^{-r(T-t)} F' N(f_2) - M\lambda' \left[k' V_t N(d_1) - e^{-r(T-t)}\left(k'F' + (N - N_r)X\right)N(d_2)\right] \\ \sigma_S = (V_t/S_t)\, \Delta_S \sigma \end{cases} \tag{3}$$

   with:

$$\Delta_S = \frac{N(f_1) - \frac{k'M}{N - N_r + k'M} N(d_1)}{N - N_r} \tag{4}$$

and where:

$$\lambda' = \frac{1}{N - N_r + k'M} \tag{5}$$

$$f_1 = \frac{\ln \frac{V_t}{F'} + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}} \tag{6}$$

$$f_2 = f_1 - \sigma\sqrt{T - t} \tag{7}$$

$$d_1 = \frac{\ln \frac{k'V_t}{k'F' + (N - N_r)X} + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}} \tag{8}$$

$$d_2 = d_1 - \sigma\sqrt{T - t} \tag{9}$$

2. Once we obtain (S_t^*, σ^*), the warrant price at time t is obtained as:

$$w(V_t,\ \sigma^*,\ X) = \frac{1}{N - N_r + k'M}\left[k'V_t N(d_1) - e^{-r(T-t)}\left(k'F' + (N - N_r)X\right)N(d_2)\right] \tag{10}$$

Table 18 shows an example of the effect of debt on the prices of warrants. We observe that, assuming a constant firm value, V_t, and a constant stock-return volatility, σ_S, the higher the face value, the lower the warrant price. In this table we also provide information about the number of stocks that the firm reduces and about the new ratio of the warrant. The last two columns show the stock price and the value of σ after the change in the debt of the firm.

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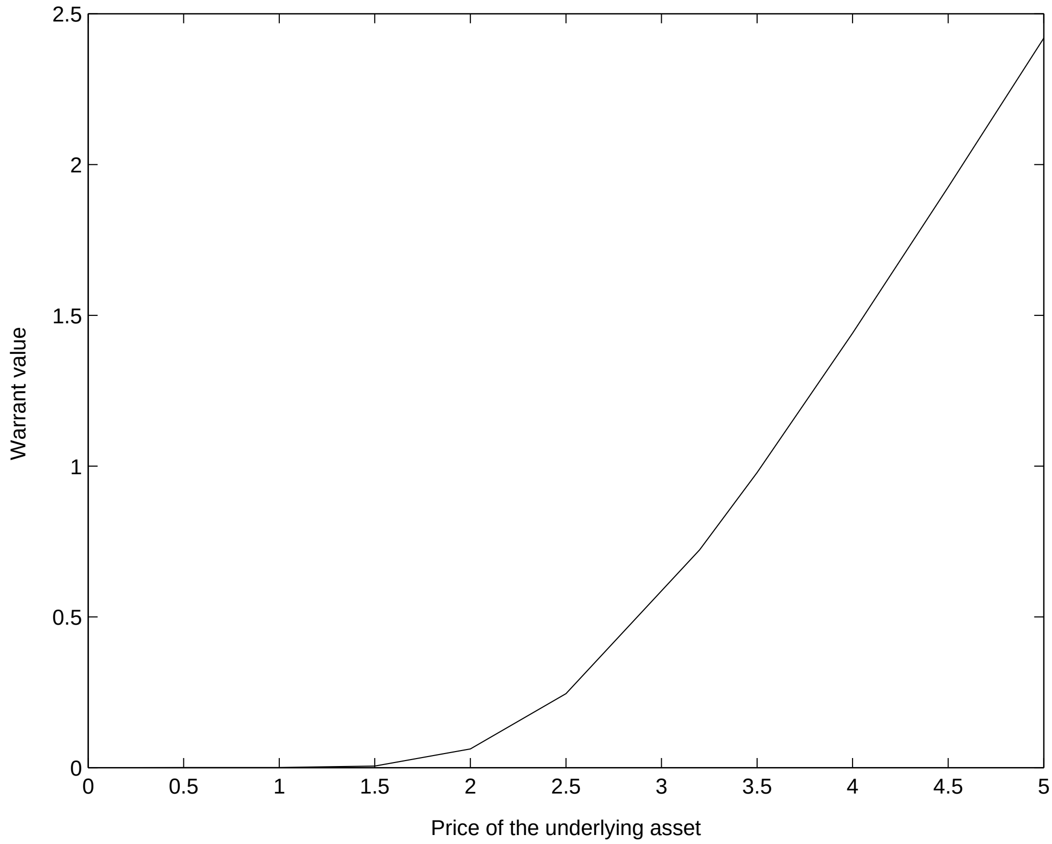

Figure 1: Warrant pricing taking into account the effect of dilution and the default risk of the issuer. The pricing model used is the extension of Ukhov's model developed in this paper. The company has 100 warrants, 10,000 shares of stock and one zero-coupon bond with face value $F = 2000$ and maturity in 2 years. Each warrant entitles the owner to receive 1 share of stock upon the payment of 3 dollars. The risk-free rate is $r = 0.04$, the volatility of the stock return is $\sigma_S = 0.2$ and the time to maturity is $T - t = 2$.

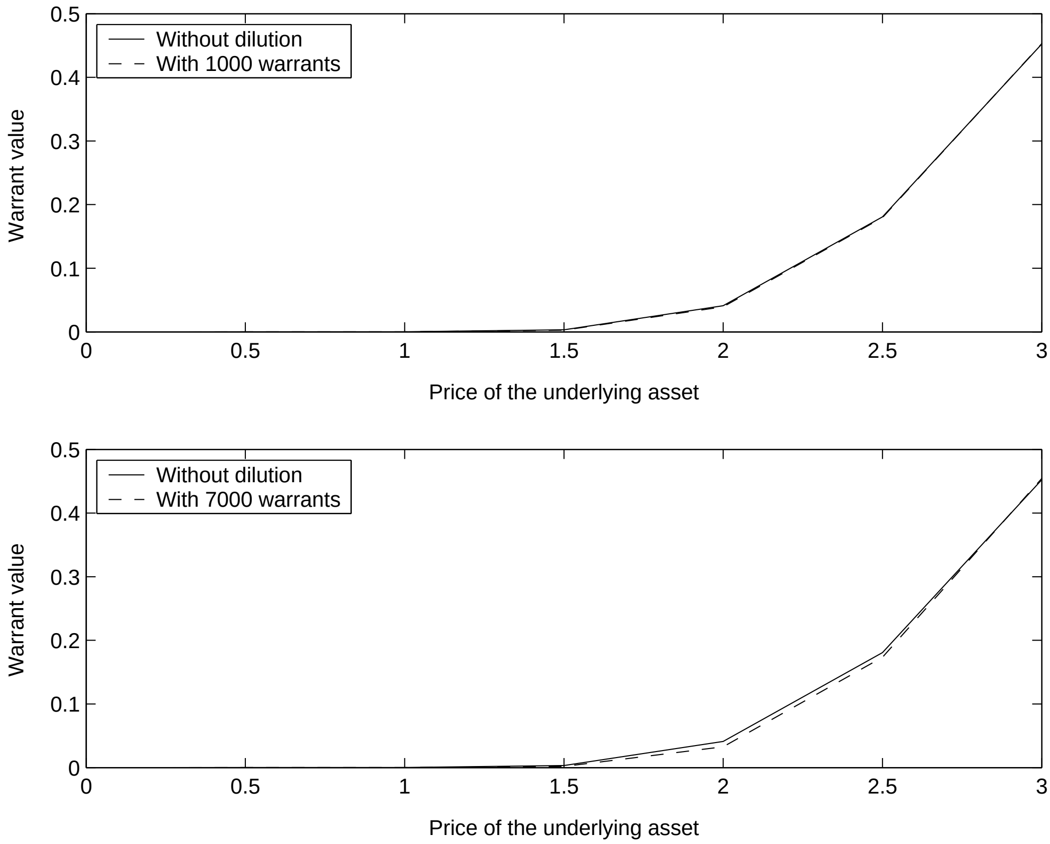

Figure 2: Effect of dilution on the price of a call warrant issued by a company without debt. We use the extension of Ukhov (2004) developed in this work. The first graph compares the value of a warrant when there is no dilution with the value of a warrant issued by a company that has $1,000$ warrants. In the graph below we compare the value of a warrant when there is no dilution with the value of a warrant issued by a company that has 7,000 warrants. In both graphs the number of stocks is $N = 10{,}000$. Each warrant entitles the owner to receive 1 unit of the underlying stock upon the payment of 3 dollars. The risk-free rate is $r = 0.04$, the volatility of the stock return is $\sigma_S = 0.2$ and the time to maturity is $T - t = 2$.

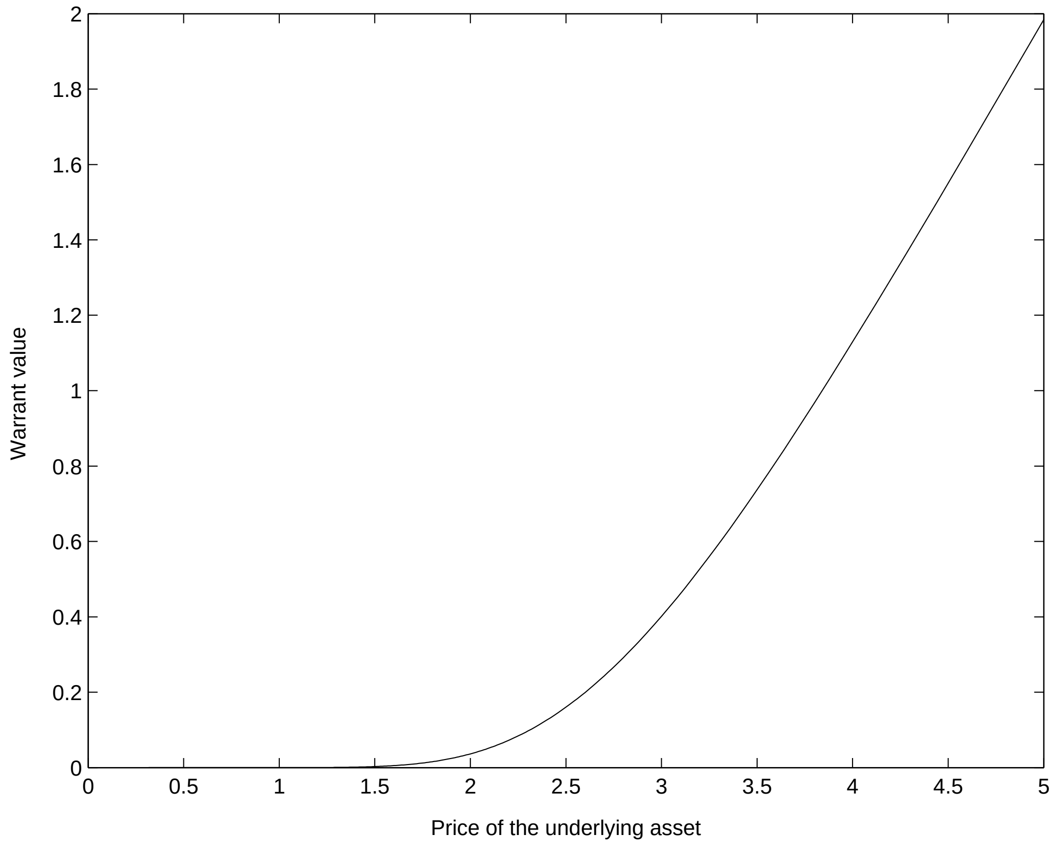

Figure 3: Pricing of warrants when there is no dilution and the credit risk of the issuer is taken into consideration. We apply the pricing model for vulnerable options proposed by Hull and White (1995). Each warrant entitles the owner to receive 1 unit of the underlying asset upon the payment of 3 dollars. The risk-free rate is $r = 0.04$ and the yield to maturity of the debt issued by the warrant issuer is $y = 0.10$. The volatility of the return of the underlying asset is $\sigma_S = 0.2$ and the time to maturity is $T - t = 2$.

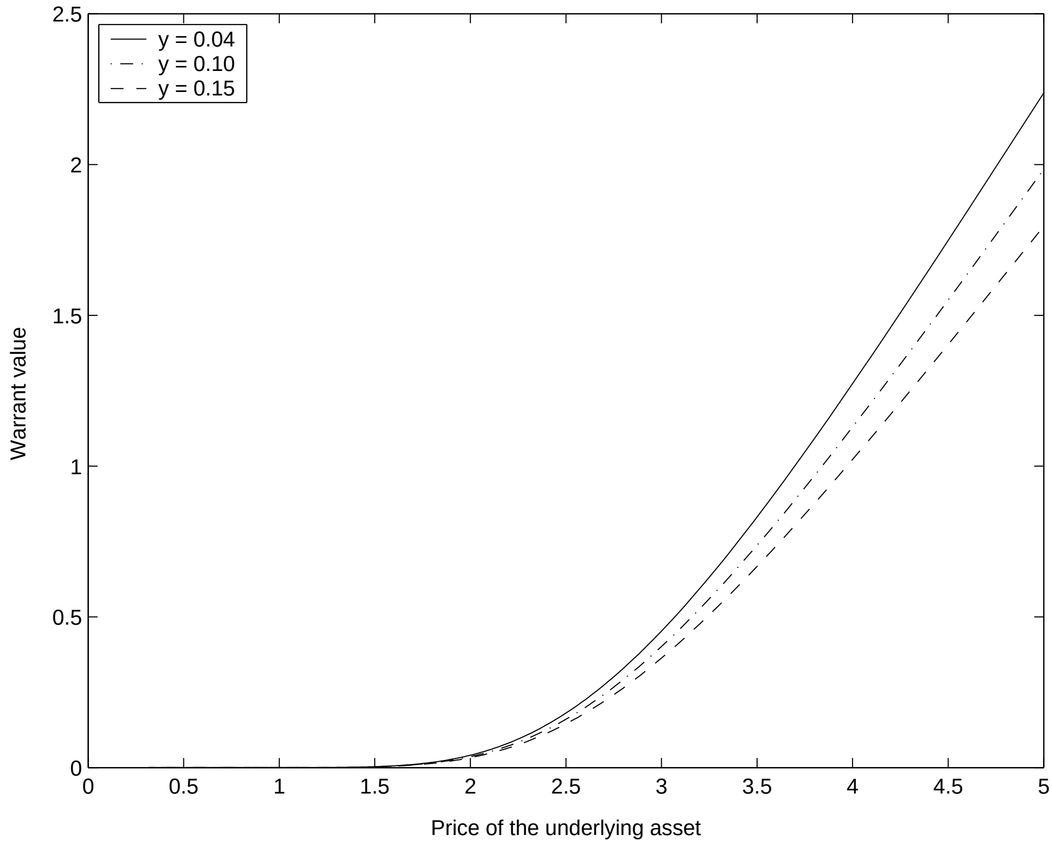

Figure 4: Effect of the issuer's credit risk on the price of a warrant without dilution. The pricing model used is the model of Hull and White (1995). Each warrant entitles the owner to receive 0.5 units of the underlying asset upon the payment of 3 dollars. The volatility of the return of the underlying asset is $\sigma_S = 0.2$ and the time to maturity is $T - t = 2$. The risk-free rate is $r = 0.04$. The results for different yields of the risky debt of the issuer are shown: $y = 0.15$, $y = 0.10$ and $y = 0.04$.

Underlying asset	Maturity	Strike price	Ratio	Issuer	Closing price
Altadis	03/17/06	32.00	0.5	BBVA	2.43
	03/17/06	32.00	0.5	BSCH	2.42
Banco	03/17/06	10.00	1	BBVA	0.57
Popular	03/17/06	10.00	1	BSCH	0.48
	03/17/06	10.40	1	BBVA	0.37
	03/17/06	10.40	1	BSCH	0.28

Table 1: Price difference between warrants with same underlying asset, maturity, strike price and ratio but issued by different companies. The pricing date is December 16, 2005.

	Rating agency		
Issuer	**Fitch**	**Moody's**	**S & P**
BBVA	AA- (April 2004)	Aa2 (August 2004)	AA- (December 2003)
BSCH	AA- (November 2003)	Aa3 (November 2003)	AA- (July 2005)
Banesto	AA- (May 2005)	Aa3 (August 2004)	A+ (August 2005)
Bankinter	A+ (October 2004)	Aa3 (November 2003)	A (April 2004)
Commerzbank	A- (June 2005)	A2 (December 2002)	A- (October 2002)
Société Générale	AA- (May 2003)	Aa2 (January 2005)	AA- (July 2005)
Sogecable	-	-	-

Table 2: Rating of the companies issuing warrants in the Spanish market according to the main rating agencies. The date of the last rating report for each issuer is indicated. We must note that there is no credit rating report for Sogecable.

Rating agency			
Standard and Poor's	**Moody's**	**Fitch**	**Financial strength**
AAA	Aaa	AAA	Exceptionally strong
AA+	Aa1	AA+	Very strong
AA	Aa2	AA	
AA-	Aa3	AA-	
A+	A1	A+	Strong
A	A2	A	
A-	A3	A-	
BBB+	Baa1	BBB+	Good
BBB	Baa2	BBB	
BBB-	Baa3	BBB-	
BB+	Ba1	BB+	Moderately weak
BB	Ba1	BB	
BB-	Ba3	BB-	
B+	B1	B+	Weak
B	B2	B	
B-	B3	B-	
CCC+	Caa1	CCC	Very weak
CCC	Caa2	CC	
CCC-	Caa3	C	
-	Ca	DDD	Distressed
-	C	DD	
-	-	D	

Table 3: Rating system of the main agencies to provide an opinion about the financial strength and ability of a company to meet its ongoing obligations.

	Underlying asset	**Call/Put**	**Type of warrant**
Warrants on Spanish indexes	FTSE Latibex Top	*Call*	American
	IBEX 35	*Call/Put*	American
Warrants on international indexes	DJ Euro Stoxx 50	*Call/Put*	American
	DJ Industrial Average	*Call/Put*	American
	Nasdaq 100	*Call/Put*	American
	Nikkei 225	*Call/Put*	American
	Xetra Dax Index 5200	*Call/Put*	American
Warrants on Spanish stocks	Abertis	*Call/Put*	American
	Altadis	*Call/Put*	American
	Amadeus	*Call/Put*	American
	B. Popular	*Call/Put*	American
	BSCH	*Call/Put*	American
	BBVA	*Call/Put*	American
	Endesa	*Call/Put*	American
	Gas Natural	*Call/Put*	American
	Iberdrola	*Call/Put*	American
	Inditex	*Call/Put*	American
	Indra	*Call/Put*	American
	Repsol	*Call/Put*	American
	Sogecable	*Call/Put*	American
	Telefónica	*Call/Put*	American
	Telefónica Móviles	*Call/Put*	American
	TPI	*Call/Put*	American
	Unión Fenosa	*Call/Put*	American
	Zeltia	*Call/Put*	American

Table 4: Warrants traded in the Spanish market. To be continued.

	Underlying asset	Call/Put	Type of warrant
Warrants on foreign stocks	Alcatel	*Call/Put*	American
	Arcelor	*Call/Put*	American
	Axa	*Call/Put*	American
	BNP Paribas	*Call/Put*	American
	Cisco Systems	*Call/Put*	American
	Deutsche Bank	*Call/Put*	American
	Deutsche Telekom	*Call/Put*	American
	Ericsson	*Call/Put*	American
	France Telecom	*Call/Put*	American
	Ing Groep NV	*Call/Put*	American
	KPN	*Call/Put*	American
	Microsoft	*Call/Put*	American
	Nokia	*Call/Put*	American
	Philips	*Call/Put*	American
	Total Fina Elf	*Call/Put*	American
Warrants on exchange rates	US Dollar - - Mexican peso	*Call/Put*	American
	Euro - US Dollar	*Call/Put*	American
	Euro - British pound	*Call/Put*	American
	Euro - Japanese yen	*Call/Put*	American
Warrants on commodities	Brent oil	*Call/Put*	European
Warrants on baskets	Basket of indexes	*Call*	European

Table 4: Continued.

Issuer	Specialist
Banesto Banco de Emisiones, S.A.	Banesto Bolsa S.V. S.A.
Bankinter, S.A.	Mercavalor, S.V. S.A.
BBVA Banco de Financiación, S.A.	Banco Bilbao Vizcaya Argentaria, S.A.
Commerzbank AG	Renta 4, S.V. S.A.
Santander Central Hispano Investment, S.A.	Santander Central Hispano Bolsa, S.V. S.A.
Société Générale Acceptance, N.V.	Société Générale, sucursal en España
Sogecable, S.A.	Santander Central Hispano Bolsa, S.V. S.A.

Table 5: Issuers and specialists in the Spanish warrants market.

Characteristics			Implied volatility		
Maturity	**Strike**	**Ratio**	**Proposed extension**	**Ukhov**	**Black-Scholes**
09/20/12	25.76	1.0094	0.4451	0.4969	0.1578

Table 6: Implied volatilities used in the pricing of warrants issued by Sogecable on the firm's own stock. The pricing date is December 16, 2005. The three first columns show the characteristics of the warrants of Sogecable. The last three columns indicate the implied volatilities obtained for the model proposed in this work and for the models of Ukhov (2004) and Black and Scholes (1973), respectively.

Characteristics			Prices given by			
Maturity	**Strike**	**Ratio**	**Proposed extension**	**Ukhov**	**B-S**	**Market**
09/20/12	25.76	1.0094	14.8517	15.3260	14.0130	13.70
			(1.983.900.000)	(4.983.900.000)		

Table 7: Application of the extension of Ukhov (2004) to the pricing of the warrants issued by Sogecable on the firm's own stock. The obtained results are compared with the prices given by the models of Ukhov (2004) and Black and Scholes (1973). The value of V_t^* is shown inside parenthesis. The pricing date is December 16, 2005. The price of the stock of Sogecable is $S_t = 34.25$ euros. The risk-free rate is $r = 0.0303$, that is the annualized rate of Treasury bonds with maturity in five years. The number of stocks outstanding is 133,564,631, the number of warrants is 1,260,043, and the face value of the debt is $F = 2{,}383.9$ millon euros.

Stock	Date	Payment	Stock	Date	Payment
Altadis	03/31/00	0.2300	**(Endesa)**	01/02/01	0.2400
	07/14/00	0.2700		07/02/01	0.4100
	06/01/01	0.5600		01/02/02	0.2640
	03/18/02	0.2800		07/01/02	0.4185
	06/20/02	0.3400		01/02/03	0.2640
	03/24/03	0.3100		07/01/03	0.4185
	06/23/03	0.3900		01/02/04	0.2640
	03/23/04	0.3500		07/01/04	0.4390
	06/22/04	0.4500		01/03/05	0.2720
	03/22/05	0.4000		07/01/05	0.4662
	06/21/05	0.5000	**Iberdrola**	01/03/00	0.2164
Banco	10/02/00	0.2900		07/03/00	0.2843
Popular	01/02/01	0.2930		01/02/01	0.2300
	04/03/01	0.3040		07/02/01	0.3155
	07/02/01	0.3080		01/02/02	0.2461
	10/01/01	0.3270		07/01/02	0.3388
	01/02/02	0.3320		01/02/03	0.2600
	04/01/02	0.3470		07/01/03	0.3505
	07/01/02	0.3540		01/02/04	0.2860
	10/01/02	0.3600		07/01/04	0.3868
	01/02/03	0.3650		01/03/05	0.3260
	04/01/03	0.3850		07/01/05	0.4421
	07/01/03	0.3900	**Repsol**	01/13/00	0.1609
	10/01/03	0.3950		07/13/00	0.2600
	01/02/04	0.4000		01/11/01	0.1900
	04/01/04	0.4050		07/12/01	0.3100
	07/01/04	0.4100		01/11/02	0.2100
	10/15/04	0.4300		01/22/03	0.1500
	01/12/05	0.4350		07/15/03	0.1600
	04/12/05	0.4450		01/15/04	0.2000
	07/12/05	0.0900		07/01/04	0.2000
	10/13/05	0.0902		01/11/05	0.2500
Endesa	01/03/00	0.2164		07/05/05	0.2500
	07/03/00	0.3726			

Table 8: Dates and payments of dividends made by Altadis, Banco Popular, Endesa, Iberdrola and Repsol in the period between January 1, 2000 and December 15, 2005.

Stock	Date of payment	Dividend in euros
Altadis	–	–
B. Popular	01/12/06	0.4350
Endesa	01/03/06	0.2720
Iberdrola	01/03/06	0.3260
Repsol	01/11/06	0.2500

Table 9: Estimation of the dividends to be paid by Altadis, Banco Popular, Endesa, Iberdrola and Repsol during the time to maturity of the warrants we studied. The maturity of all warrants is March 17, 2006 with the exception of Endesa that matures June 16, 2006. The second column of the table shows the estimated date of payment and the last column indicates the amount to be paid by each stock.

Underlying	Characteristics				Implied volatility	
asset	Strike	Maturity	Ratio	Issuer	With credit risk	Without credit risk
Altadis	32	03/17/06	0.5	BBVA	0.2716	0.2713
Altadis	32	03/17/06	0.5	BSCH	0.2722	0.2713
B. Popular	10.4	03/17/06	1	BBVA	0.3017	0.3017
B. Popular	10.4	03/17/06	1	BSCH	0.2558	0.2557
B. Popular	10	03/17/06	1	BBVA	0.3203	0.3203
B. Popular	10	03/17/06	1	BSCH	0.2731	0.2729
Endesa	18	06/16/06	0.5	BBVA	0.2695	0.2695
Endesa	18	06/16/06	1	BSCH	0.2737	0.2723
Iberdrola	21	03/17/06	0.5	BBVA	0.2118	0.2117
Iberdrola	21	03/17/06	1	BSCH	0.2510	0.2506
Repsol	21	03/17/06	0.5	BBVA	0.3200	0.3195
Repsol	21	03/17/06	1	BSCH	0.3366	0.3352

Table 10: Implied volatilities used in the pricing of warrants on December 16, 2005. The first column indicates the stock underlying the warrant. The next four columns show the characteristics of each warrant. Finally, the two last columns provide the implied volatility obtained for the models of Hull and White (1995) and Black and Scholes (1973), respectively.

		Characteristics				Prices given by		
Issuer	**Maturity**	**Strike**	**Moneyness**		**Ratio**	**H-W**	**B-S**	**Market**
BBVA	03/17/06	32	0.8888	*Deep ITM*	0.5	2.3052	2.3051	2.43
BSCH	03/17/06	32	0.8888	*Deep ITM*	0.5	2.3051	2.3051	2.42

Table 11: Application of the model of Hull and White (1995) to the pricing of warrants on Altadis. The pricing date is December 16, 2005. The price of Altadis is $S_t = 36$ euros. The risk-free rate is $r = 0.0207$, that is the annualized rate of Treasury bonds with maturity in three months. The yield to maturity for the debt issued by BBVA is $y = 0.0217$, and for the debt issued by BSCH is $y = 0.024273$.

		Characteristics				Prices given by		
Issuer	**Maturity**	**Strike**	**Moneyness**		**Ratio**	**H-W**	**B-S**	**Market**
BBVA	03/17/06	10	0.9775	*ITM*	1	0.5519	0.5520	0.57
BSCH	03/17/06	10	0.9775	*ITM*	1	0.4601	0.4601	0.48
BBVA	03/17/06	10.4	1.0166	*OTM*	1	0.3652	0.3653	0.37
BSCH	03/17/06	10.4	1.0166	*OTM*	1	0.2807	0.2807	0.28

Table 12: Application of the model of Hull and White (1995) to the pricing of warrants on Banco Popular. The pricing date is December 16, 2005. The price of Banco Popular is $S_t = 10.23$ euros. The risk-free rate is $r = 0.0207$, that is the annualized rate of Treasury bonds with maturity in three months. The yield to maturity for the debt issued by BBVA is $y = 0.0217$, and for the debt issued by BSCH is $y = 0.024273$.

		Characteristics				Prices given by		
Issuer	**Maturity**	**Strike**	**Moneyness**		**Ratio**	**H-W**	**B-S**	**Market**
BBVA	06/16/06	18	0.8275	*Deep ITM*	0.5	1.9933	1.9933	1.98
BSCH	06/16/06	18	0.8275	*Deep ITM*	1	3.9957	3.9961	3.99

Table 13: Application of the model of Hull and White (1995) to the pricing of warrants on Endesa. The pricing date is December 16, 2005. The price of Endesa is $S_t = 21.75$ euros. The risk-free rate is $r = 0.0207$, that is the annualized rate of Treasury bonds with maturity in three months. The yield to maturity for the debt issued by BBVA is $y = 0.0217$, and for the debt issued by BSCH is $y = 0.024273$.

Characteristics						Prices given by		
Issuer	**Maturity**	**Strike**	**Moneyness**		**Ratio**	**H-W**	**B-S**	**Market**
BBVA	03/17/06	21	0.9341	*Deep ITM*	0.5	0.8347	0.8347	0.83
BSCH	03/17/06	21	0.9341	*Deep ITM*	1	1.8145	1.8146	1.81

Table 14: Application of the model of Hull and White (1995) to the pricing of warrants on Iberdrola. The pricing date is December 16, 2005. The price of Iberdrola is $S_t = 22.48$ euros. The risk-free rate is $r = 0.0207$, that is the annualized rate of Treasury bonds with maturity in three months. The yield to maturity for the debt issued by BBVA is $y = 0.0217$, and for the debt issued by BSCH is $y = 0.024273$.

Characteristics						Prices given by		
Issuer	**Maturity**	**Strike**	**Moneyness**		**Ratio**	**H-W**	**B-S**	**Market**
BBVA	03/17/06	21	0.8353	*Deep ITM*	0.5	2.1218	2.1217	2.17
BSCH	03/17/06	21	0.8353	*Deep ITM*	1	4.2828	4.2829	4.37

Table 15: Application of the model of Hull and White (1995) to the pricing of warrants on Repsol. The pricing date is December 16, 2005. The price of Repsol is $S_t = 25.14$ euros. The risk-free rate is $r = 0.0207$, that is the annualized rate of Treasury bonds with maturity in three months. The yield to maturity for the debt issued by BBVA is $y = 0.0217$, and for the debt issued by BSCH is $y = 0.024273$.

Characteristics				Implied volatility	
Maturity	**Strike**	**Ratio**	**Issuer**	**With credit risk**	**Without credit risk**
06/16/06	10,500	0.001	BBVA	0.1498	0.1498
06/16/06	10,500	0.002	Banesto	0.1963	0.1959
03/17/06	10,000	0.001	BBVA	0.1536	0.1535
03/17/06	10,000	0.002	BSCH	0.1570	0.1566

Table 16: Implied volatilities used in the pricing of warrants on the index IBEX-35. The pricing date is December 16, 2005. The first four columns show the characteristics of each warrant. The last two columns offer the implied volatility obtained for the models of Hull and White (1995) and Black and Scholes (1973), respectively.

Characteristics						Prices given by		
Issuer	Maturity	Strike	Moneyness		Ratio	H-W	B-S	Market
BBVA	03/17/06	10,000	0.9473	*Deep ITM*	0.001	0.7041	0.7041	0.73
BSCH	03/17/06	10,000	0.9473	*Deep ITM*	0.002	1.4175	1.4176	1.47
BBVA	06/16/06	10,500	0.9947	*ATM*	0.001	0.5294	0.5294	0.55
Banesto	06/16/06	10,500	0.9947	*ATM*	0.002	1.3273	1.3271	1.34

Table 17: Application of the model of Hull and White (1995) to the pricing of warrants on the index IBEX-35. The pricing date is December 16, 2005. The value of IBEX-35 is $S_t = 10{,}555.30$. The risk-free rate is $r = 0.0207$ for the warrants with maturity in March 2006 and $r = 0.0217$ for the warrants with maturity in June 2006. The yield to maturity for the debt issued by BBVA is $y = 0.0217$, for the debt issued by Banesto is $y = 0.024898$, and for the debt issued by BSCH is $y = 0.024273$.

Face value of debt	Warrant price	Reduction of the number of stocks	Modified ratio	Stock price after	Volatility after
3,000	0.0386	0.00	1.0000	2.0004	0.1759
4,000	0.0255	461.47	0.9539	2.0005	0.1677
5,000	0.0158	922.93	0.9077	2.0007	0.1596
8,000	0.0022	2307.30	0.7693	2.0008	0.1352

Table 18: Example of the effect of debt on the prices of warrants issued on a firm's own stocks. The firm has $10,000$ shares of stock with a price of 2.0004 dollars each, 100 European call warrants with 2 year maturity and strike price 3 dollars, and 1 zero-coupon with face value of 3,000 dollars and 2 year maturity. The firm value is 22,777 dollars, the risk-free rate is 0.04 and the stock return standard deviation is 0.2. The first two columns show the effect of the face value on the price of the warrant. The next two columns indicate the number of own stocks that are bought by the company and the new ratio of the warrants. Finally, the last two columns show the stock price and the value of σ for each level of debt leverage.