A SINTHESIS OF LOCATION MODELS

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Abstract: This article considers a model of spatial competition where firms and consumers are located in a semicircular space rather than in the whole circle (Salop's model) or the linear city (Hotelling's model), under the assumptions of both, convex and concave, transportation costs.

The paper tries to generalize the results of the two previous models. We find that the existence of a price equilibrium is warranted for every firms' location when the length of the semicircular space is greater than 3/4.

Resumen: El artículo considera un modelo de competencia espacial donde las empresas y los consumidores se localizan en un espacio semi-circular en lugar del círculo completo (modelo de Salop) o la ciudad lineal (modelo de Hotelling), bajo el supuesto de costes de transporte cuadráticos.

El trabajo intenta generalizar los resultados de los dos modelos anteriores. Encontramos que la existencia de un equilibrio en precios está garantizada para cualquier localización de las empresas cuando la longitud del espacio semi-circular es mayor que 3/4.

Key words: urban design, regulator, spatial competition, transportation costs, segmented market

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1.-Introduction

In most urban configurations, we observe the existence of portions of land devoted to non-residential purposes. We can find environmentally protected areas, parks, recreational facilities, etc. When considering urban design, regulators have to decide whether they should leave some part of urban land for this type of recreational activities, and if they do so, what is the optimal size of these non-residential areas.

These zones represent a discontinuity in the market since neither consumers nor firms can be located within them; however, it is possible to find consumers and firms adjacent to both ends of these restricted areas.

In order to study the implications of this type of urban configurations, we consider a circular spatial model, where there is a segment where people live and firms locate, that we shall refer to as the market, and another segment that can only be used for non-residential purposes. (See figure 1)

In this context, a three-stage game can be considered in which in the first stage the regulator chooses the size of the market, in the second stage firms choose locations and in the third stage firms compete in prices.

We will consider that the regulator chooses the length of the market in a non-strategic manner; therefore, the model we present, once market size is given, can be reduced to a two-stage game in which firms first choose location, and then compete in prices.

This model can be thought of as a synthesis of the circular and linear spatial models. If the size of the market is half of the circumference, $l/2$, or less, the model is equivalent to Hotelling's (1929) linear city, while if the size of the market is the whole circle, l, we are in Salop's (1979) configuration. We analyze the existence of a price equilibrium when the market segment, h, is restricted to be less than l.

Together with market space configuration, industry structure, demand distribution and the type of transportation costs function assumed are fundamental in defining a spatial model, and these choices will determine the existence or non-existence of equilibrium.

The so-called existence problem of equilibrium has produced a large number of academic works in which different assumptions about number of firms, consumer's densities, transportation costs functions, etc., have been made. For a revision of the seminal papers of this literature see Gabszewicz and Thisse (1986) and the references cited within it.

As mentioned above, an important line of research has dealt with the type of transportation costs functions that can ensure equilibrium existence in both, the linear and circular models. In the linear model, D'Aspremont et al. (1979) used a quadratic transportation costs function and showed the existence of a maximum differentiation equilibrium. Economides (1986) studied the function $C(d) = d^{a}$, $1 \leq a \leq 2$ and determined the values of a for which price equilibrium exists. Gabszewicz and Thisse (1986) and Anderson (1988) have studied the transportation costs functions $C(d) = ad+bd^2$, $a>0$, $b>0$, and found that a perfect equilibrium does not exist for every possible firms' location; while Hamoudi & Moral (2005) showed that, with concave transportation costs, equilibrium cannot be attained.

In the circular model, Anderson (1986) studied equilibrium existence for convex transportation costs functions. De Frutos et al. (1999) have analyzed convex and concave transportation costs functions and they have proven the existence of equilibrium for any location of firms for the functions: $C(d) = k(d-d^2)$ and $C'(d) = kd^2$.

We will make the standard assumptions of two firms selling a homogeneous product, consumers evenly distributed a long the market segment, and we will study the existence problem in our model using the two linear quadratic functions that ensure the existence of a perfect equilibrium in pure strategies in the circular model.

We find that there exists a price equilibrium for any location of firms provided that the length of the market segment is greater than approximately $3/4$. Furthermore, this equilibrium is unique and implies firms locating opposite to each other. However, unlike the results obtained in the circular model, the equilibrium is not symmetric since one of the firms endures a competitive disadvantage due to the market discontinuity. If h is smaller than $3/4$, we find that, for certain values of h and firms locations, there is a strip where no price equilibrium may exist. Nevertheless, there are many combinations of market sizes and firm locations for which equilibrium can be obtained.

When we compare the intensity of competition, given the size of the market, we find that competition is more intense for low values of *h*, and the equilibrium region is smaller.

The paper is organized as follows, in section 2 we present the model under the hypothesis of both concave and convex transportation costs, in section 3 we study the existence of equilibrium, section 4 contains the conclusions, and finally major proofs and graphs can be found in the appendix.

2.- The Model

We consider a circular city of length l where the regulator chooses the size of the market, *h*, so that firms and consumers can only be located on a certain segment of the circle $h \leq l$. There are two firms selling a homogeneous product, with zero production costs, located at x and y, with $0 \leq x \leq y \leq h$.

Consumers are evenly distributed along *h*, and each consumer buys a single unit of this product per unit of time, irrespective of its price. Since the product is homogeneous, consumers will buy from the firm who offers the least delivered price, that is, the mill price plus transportation costs. Let p_1, p_2, denote the mill prices charged by firms located at x and y, respectively. The distance between consumer z and firm i is given by $d_i = |z - i|$, $i = x, y$. We will consider two possible types of transportation costs: concave and convex. In particular, we will assume the only two cost functions from the linear quadratic family: $C(d_i) = k(d_i\text{-}d_i^2)$ and $C'(d_i) = kd_i^2$, that have been shown to ensure existence of a perfect equilibrium in pure strategies in the circular model. (See De frutos et all., 1999). These two cost functions are strategically equivalent, and therefore, the equilibrium results obtained in one case can be immediately translated to the other by an appropriate change of variables, (See De Frutos et al, 2002)

Although firms and consumers can only be located within h, consumers can travel along the whole circle and they will always take the direction that implies the shorter distance to the chosen firm.

The model described above, given that the regulator behaves in a non-strategic manner, does give rise to a two-stage game in which firms first decide simultaneously their location and then simultaneously choose prices. It turns out that the solution to this game depends critically on the length of the market segment, h.

In order to determine the market boundaries and derive the demands faced by each firm, we will have to find the indifferent consumers. A consumer is indifferent to buying from one firm or the other if and only if:

$p_1 + C(d_1) = p_2 + C(d_2)$.

That is, a consumer is indifferent if she faces the same full price from the two firms.

We find that, depending on the length of h considered, one or two indifferent consumers may exist simultaneously, and different price intervals, I_j^i, for which demand is defined, will be obtained. We will only present the demand for firm 1, since demand for firm 2, is always given by:

$D_2 = h - D_1$

2.1.-Concave Transportation Costs

The transportation costs function faced by consumers is given by:
$C(d_i) = k(d_i\text{-}d_i^2)$. We assume, without loss of generality that $l = k = 1$, $x = 0$ and $0 \le y \le 1/2$.

When the market length considered is the whole circle (See figure 1), four possible indifferent consumers are found, each one belonging to a different segment of the circumference given by: $m_1 \in [0, y]$, $m_2 \in [y, ½]$, $m_2' \in [½, y+½]$, $m_2'' \in [y+½, 1]$, however $m_2 = m_2' = m_2''$. Therefore, only two distinct indifferent consumers may exist: an indifferent consumer $m_1 \in [0, y]$ and, an indifferent consumer, that we will denote, m_2, $m_2 \in [y, 1]$, given by the following expressions:

$$m_1 = \frac{p_2 - p_1}{2(1-y)} + \frac{y}{2} \qquad (1)$$

$$m_2 = \frac{p_1 - p_2}{2y} + \frac{y}{2} + \frac{1}{2} \qquad (2)$$

Both belonging to the prices interval: $-y(1-y) \leq p_1 - p_2 \leq y(1-y)$

If we restrict the length of the market to $h, y \leq h \leq 1$, there are two distinct cases to consider:

Either $y \leq m_2 \leq h$, and there exist two indifferent consumers, $m_1 \in [0, y]$, belonging to the full prices interval, as above, and an indifferent consumer $m_2 \in [y, h]$, belonging to the price interval $-y(1-y) \leq p_1 - p_2 \leq y(2h - 1 - y)$.

On the other hand, if $h \leq m_2 \leq 1$, there is only one indifferent consumer, m_1, for the prices interval $y(2h - 1 - y) \leq p_1 - p_2 \leq y(1-y)$.

Every consumer $m \leq m_1$ and every consumer $m \geq m_2$ will buy from firm 1.

Demand for firm 1 is given by:

$$D_1 = \begin{cases} h & \text{for} \quad p_1 - p_2 \in I_1^1 \\ m_1 + (h - m_2) & \text{for} \quad p_1 - p_2 \in I_2^1 \\ m_1 & \text{for} \quad p_1 - p_2 \in I_3^1 \\ 0 & \text{for} \quad p_1 - p_2 \in I_4^1 \end{cases}$$

Where:

$I_1^1 = [-\infty, -y(1-y)]$, $I_2^1 = [-y(1-y), y(2h - y - 1)]$, $I_3^1 = [y(2h - y - 1), y(1-y)]$, $I_4^1 = [y(1-y), +\infty]$

2.2.- Convex Transportation Costs

As has been shown in De Frutos et all., (2002), in the circular model, for any convex transport cost function there exists a concave one such that the location-then-price games induced by these functions are strategically equivalent. In particular, they show that in a circular market, for any consumer m facing a concave transportation cost function, there is another consumer n facing a convex transportation cost function whose utility is a monotone transformation of the utility of m, and therefore, n will buy to the same seller as m. We use this result, and an appropriate transformation of variables (see figure 2), to study the convex transportation costs case.

Let $C(d_i) = kd_i^2$ and that $l = k = 1$, $x < y$ and $y = 1/2$. In this case, and for the whole circular market we obtained two types of indifferent consumers: $n_1 = n_1' = n_1'' \in [0, x + \frac{1}{2}]$ and $n_2 \in [x + \frac{1}{2}, 1]$, given by:

$$n_1 = \frac{p_2 - p_1}{2(\frac{1}{2} - x)} + \frac{1}{2}(x + \frac{1}{2}) \qquad (3)$$

$$n_2 = \frac{p_1 - p_2}{2(\frac{1}{2} + x)} + \frac{1}{2}(\frac{1}{2} + x) + \frac{1}{2} \qquad (4)$$

Both defined for the prices interval: $-(\frac{1}{2} - x)(\frac{1}{2} + x) \le p_1 - p_2 \le (\frac{1}{2} - x)(\frac{1}{2} + x)$.
Every consumer $n \le n_1$ and every consumer $n \ge n_2$ will buy from firm 1.

When we restrict the length of the market to $y \le h \le 1$, Either $n_2 \le h$, and there are two indifferent consumers, $n_1 \in [0, x + \frac{1}{2}]$, defined for the full prices interval, as above, and $n_2 \in [x + \frac{1}{2}, h]$, belonging to the prices interval $[-(\frac{1}{2} - x)(\frac{1}{2} + x) \le p_1 - p_2 \le (\frac{1}{2} + x)(2h - (\frac{1}{2} + x) - 1)]$,
or $h \le n_2$ and there is only one indifferent consumer, n_1, defined in $[(\frac{1}{2} + x)(2h - (\frac{1}{2} + x) - 1) \le p_1 - p_2 \le (\frac{1}{2} - x)(\frac{1}{2} + x)]$
Demand for firm 1 is given by:

$$D_1 = \begin{cases} h & for \quad p_1 - p_2 \in I_1^2 \\ n_1 + (h - n_2) & for \quad p_1 - p_2 \in I_2^2 \\ n_1 & for \quad p_1 - p_2 \in I_3^2 \\ 0 & for \quad p_1 - p_2 \in I_4^2 \end{cases}$$

Where:

$I_1^2 = \left[-\infty, -(1/2 - x)(1/2 + x)\right]$, $I_2^2 = \left[-(1/2 - x)(1/2 + x), (1/2 + x)(2h - (1/2 + x) - 1)\right]$

$I_3^2 = \left[(1/2 + x)(2h - (1/2 + x) - 1), (1/2 - x)(1/2 + x)\right]$, $I_4^2 = \left[(1/2 - x)(1/2 + x), +\infty\right]$

Note that the results obtained are identical to the concave case for $y = (1/2 + x)$. Since the demand functions are identical to those of the concave case, we will only have to study equilibrium for one of the cost functions.

3.- Equilibrium

Given the size of the market, *h*, and using the usual approach for a two-stage non-cooperative game in which firms select a position at the first stage and subsequently set their prices, we study the subgame perfect equilibrium. We recall that a perfect price-location equilibrium is defined as a pair $\left(p_1^N, 0\right)$, $\left(p_2^N, y^N\right)$ such that:

(i) $p_1^N = p_1^N\left(0, y^N, h\right)$ and $p_2^N = p_2^N\left(0, y^N, h\right)$,

(ii) $B_2(0, y^N, h, p_1^N(0, y^N, h), p_2^N(0, y^N, h)) \geq B_2(0, y, h, p_1^N(0, y^N, h), p_2^N(0, y^N, h))$

$\forall y \in [0, 1/2]$.

Where (p_1^N, p_2^N) is a Nash equilibrium in the price subgame when the locations choice is fixed.

We will now compute the equilibrium of the price subgame. A Nash equilibrium in the price subgame will be a pair of prices (p_1^N, p_2^N), such that :

$$p_1^N = \underset{P_1}{ArgMax} B_1\left(p_1, p_2^N\right), \quad p_2^N = \underset{P_2}{ArgMax} B_1\left(p_1^N, p_2\right)$$

Since production costs are assumed to be zero, the profit functions for firm *i* are given by $B_i = p_i D_i$, $i = 1,2$. Therefore, the expressions for the profit functions for firms 1 and 2 are given by:

$$B_1 = \begin{cases} hp_1 & for \quad p_1 - p_2 \in I_1^1 \\ p_1\left(\dfrac{p_2 - p_1}{2y(1-y)} + \dfrac{(2h-1)}{2} \right) & for \quad p_1 - p_2 \in I_2^1 \\ p_1\left(\dfrac{p_1 - p_2}{2(1-y)} + \dfrac{y}{2} \right) & for \quad p_1 - p_2 \in I_3^1 \\ 0 & for \quad p_1 - p_2 \in I_4^1 \end{cases} \tag{5}$$

$$B_2 = \begin{cases} 0 & for \quad p_1 - p_2 \in I_1^1 \\ p_2\left(\dfrac{p_1 - p_2}{2(1-y)} + \dfrac{y}{2} \right) & for \quad p_1 - p_2 \in I_2^1 \\ p_2\left(\dfrac{p_2 - p_1}{2y(1-y)} + \dfrac{(2h-1)}{2} \right) & for \quad p_1 - p_2 \in I_3^1 \\ hp_2 & for \quad p_1 - p_2 \in I_4^1 \end{cases} \tag{6}$$

If we look at the profit function of firm 1, it is obvious that no price equilibrium can exist in region I_4^1, since firm 1 could always benefit from lowering its price and capturing some demand from firm 2. The same reasoning applies to firm 2 in region I_1^1, therefore, in order to explore possible equilibria we can restrict ourselves to regions I_2^1 and I_3^1. However, the profit functions are not concave in prices and they could exhibit different configurations. In particular, they may exhibit one or two local maxima. Depending on the values of y two cases may arise, one in which the global optimum belongs to region I_2^1, and another in which the global optimum belongs to I_3^1.

3.1.- Equilibrium in Region I_2^1

For this prices interval two indifferent consumers exist.

Computing the first order conditions for the profit functions in I_2^1, we obtain:

$$p_1^{N1} = \frac{1}{3} y(1-y)(4h-1) \quad \text{and} \quad p_2^{N1} = \frac{1}{3} y(1-y)(2h+1) \tag{7}$$

and profits are given by:

$B_1(p_1^{N1}, p_2^{N1}) = \dfrac{1}{18} y(1-y)(4h-1)^2$ and

$$B_2(p_1^{N1}, p_2^{N1}) = \frac{1}{18} y(1-y)(2h+1)^2 \qquad (8)$$

Proposition 1: The pair $(p_1^{N1}(0,y,h), p_2^{N1}(0,y,h))$ constitutes a Nash equilibrium if and only if $(h,y) \in R_1$, where: $R_1 = \left\{ (h,y) \;\middle|\; h \geq h_{12}, where, h_{12}(y) = \frac{1+2\sqrt{y}}{4-\sqrt{y}} \right\}$

Proof: See Appendix

Proposition 2: There exists a subgame perfect price-location equilibrium if and only if $h \geq 0{,}733$ and then, this equilibrium is unique and is given by:

$$y^N = \frac{1}{2}, \quad p_1^N(0, y^N, h) = \frac{4h-1}{12}, \quad p_2^N(0, y^N, h) = \frac{2h+1}{12}$$

Proof:

From proposition 1, there exists a price equilibrium if $h \geq h_{12}(y)$, however, simple calculations show that $h_{12}(y)$ is increasing ($\frac{\partial h_{12}}{\partial y} = \frac{9}{2\sqrt{y}(4-\sqrt{y})^2} > 0$) and reaches a maximum for $y = \frac{1}{2}$ (see figure 4) and $h_{12}(\frac{1}{2}) = \frac{\sqrt{2}+2}{4\sqrt{2}-1} \approx 0{,}733$, therefore $\forall h \geq 0{,}733$ there exist a price equilibrium for any firms location.

If we look at the optimal location of firm 2 (given that the location of firm 1 is fixed at 0) we find that $\frac{\partial B_2}{\partial y} = 0 \quad \Rightarrow y = y^N = \frac{1}{2}$. Substituting y by $\frac{1}{2}$ in the expressions of $p_1^{N1}(0,y,h)$ and $p_2^{N1}(0,y,h)$ we obtain $p_1^N(0, y^N, h) = \frac{4h-1}{12}$, $p_2^N(0, y^N, h) = \frac{2h+1}{12}$.?

3.2.- Equilibrium in Region I_3^1

For this prices interval only one indifferent consumer exists.

Computing the first order conditions for the profit functions in I_3^1, we obtain:

$$p_1^{N2} = \frac{1}{3}(1-y)(2h+1) \qquad \text{and} \qquad p_2^{N2} = \frac{1}{3}(1-y)(4h-1) \qquad (9)$$

and profits are given by:

$$B_1(p_1^{N2}, p_2^{N2}) = \frac{1}{18}(1-y)(2h+y)^2 \text{ and}$$

$$B_2(p_1^{N2}, p_2^{N2}) = \frac{1}{18}(1-y)(4h-1)^2 \qquad (10)$$

Proposition 3: The pair $(p_1^{N2}(0,y,h), p_2^{N2}(0,y,h))$ constitutes a Nash equilibrium if and only if $(h,y)\in R_2$, Where: $R_2 = \left\{(h,y) \middle| \; h \leq h_{21}, where, h_{21} = \frac{y(5+y)}{2(2y+1)}\right\}$ (11)

Proof: See Appendix.

Corollary: if $h < 0{,}733$, there is no price equilibrium in region R defined by:

$R = \{(h,y) / h_{12}(y) < h < h_{21}(y)\}$

Proof:

R is the intersection of the two complements of regions R_1 and R_2.

Figure 6 combines the two equilibrium regions depicted in the previous two figures. As can be seen from the graph, there is only a narrow strip where no equilibrium exists.?

In figure 4, the equilibrium area for region I_2^1 is depicted, all points in the shaded area are possible equilibria. Note that for $h \geq 0{,}733$ there exists a price equilibrium for every possible location of firm 2. That is, provided that the regulator does not devote an area greater than a quarter of the city size to non-residential use, this type of urban configuration does not pose any equilibrium problems.

When we look at the optimal location of firm 2 in region I_2^1, when there are two indifferent consumers, we find that firm 2 benefits from moving away from firm 1 and locating at the opposite boundary of the market segment where m_1 exists. The result is equivalent to what we obtain when we consider the complete circular model: there are two indifferent consumers and firms locate opposite to each other and equidistant to the two indifferent consumers. However, in our model the competitive situation of the two firms is not the same. In the market segment [0, ½], where the indifferent consumer m_1 exists, the situation of the two firms is symmetric and that is why firm 2 will choose the opposite extreme to firm 1. However, when we consider the left half of the circumference, firm 1 is located at the edge of the non-residential area, and therefore this side of its potential market is restricted, as a result, firm 2 may charge a larger price and obtain larger profits than firm 1.

Figure 5 shows the equilibrium area for region I_3^1. All points belonging to the shaded area (above the line $h = y$), are equilibrium candidates.

When we look at the optimal location of firm 2 in this region, we find that $\frac{\partial B_2}{\partial y} = -\frac{1}{18}(4h-1)^2 < 0$ and therefore, firm 2 will tend to move towards firm 1. In region I_3^1 there is only one indifferent consumer and the market resembles the linear city case, given that firm 1 is fixed at 0, as firm 2 moves closer to firm 1 it squeezes the demand of firm 1.

If we look at the prices differences of the two equilibrium regions we obtain: $D^{N1} = p_1^{N1} - p_2^{N1} = \frac{-2y(1-y)(1-h)}{3} < 0$, $D^{N2} = p_1^{N2} - p_2^{N2} = \frac{-2y(1-y)(h-y)}{3} < 0$,

We can see that, in both cases, the price of firm 1 is smaller than that of firm 2, this can be explained, as mentioned above, in terms of market configuration since firm 2 has more potential customers than firm 1.

Also, when we compare the prices differences in both regions, D^{N1} and D^{N2}, we find that: $D^{N1} - D^{N2} = \frac{-2y(1-y)(h(1-y)+2y)}{3} > 0$ for all (h, y), so $D^{N1} > D^{N2}$. This implies that the intensity of competition is stronger in the second case than in the first one. This is not surprising since market size is smaller in the second case than in the first and this must induce stronger competition. This can explain why the equilibrium region of the second case is smaller than the first one.

4.- Conclusions

In most urban configurations, the existence of portions of land devoted to non-residential purposes is observed. In considering urban design, regulators have to decide what the optimal size of these non-residential areas is.

In this article we propose a circular model of spatial competition in which the market is restricted in order to allow for a non-residential area. Aside from this discontinuity in the market, we make standard assumptions and use the two linear quadratic functions that have been proven to ensure the existence of price equilibrium in the circular model.

We find that, provided the regulator chooses the size of the market segment to be greater than approximately $3/4$ $(h \geq 0{,}733)$, there is a subgame perfect price-location equilibrium and this equilibrium is unique. However, this equilibrium will not be symmetric, since the firm located adjacent to the market discontinuity will endure a competitive disadvantage. On the other hand, if market size is chosen to be less than $3/4$, for certain values of h and firms locations, there is a narrow strip where no price equilibrium may exist. As usual, equilibrium failure is due to the non-convexities exhibited by profit functions.

5.-Appendix

Proof of Proposition 1:

In order for (p_1^{N1}, p_2^{N1}) to be a price equilibrium, the following conditions must be met:

(i) $p_1^{N1} - p_2^{N1} \in I_2^1$

(ii) $B_1(p_1^{N1}, p_2^{N1}) \geq B_1(p_1, p_2^{N1}) \forall p_1$

(iii) $B_2(p_1^{N1}, p_2^{N1}) \geq B_2(p_1^{N1}, p_2) \forall p_2$

Condition (i) Implies that:

a) $-y(1-y) \leq p_1^{N1} - p_2^{N1}$ Which is always true, and

b) $p_1^{N1} - p_2^{N1} \leq y(2h-1-y)$

Which holds if and only if: $h \geq h_{11}$, where $h_{11} = \dfrac{(1+5y)}{2(2+y)}$

So (p_1^{N1}, p_2^{N1}) could be a Nash price equilibrium if y and h belong to the set:

$R_{11} = \{(h,y) | \ h \geq h_{11}\}$

In order to verify condition (ii) we have to check, for p_2^{N1} given, what is the maximum p_1^{**} for $B_1(p_1^{**}, p_2^{N1})$ in I_3^1 which is reached at:

a) $p_1^{**} = p_2^{N1} + y(2h-y-1)$, in this case we have $B_1(p_1^{N1}, p_2^{N1}) > B_1(p_1^{**}, p_2^{N1})$

b) $p_1^{**} = \dfrac{1}{3} y(1-y)(2h+1)$, Where $p_1^{**} > p_2^{N1} + y(2h-y-1)$, and the profit function for firm 1 is given by: $B_1(p_1^{**}, p_2^{N1}) = \dfrac{1}{18}(1-y)[y(2+h)\ ]^2$, some simple calculation then show that $B_1(p_1^{N1}, p_2^{N1}) \geq B_1(p_1^{**}, p_2^{N1})$ for all (h,y) such that $h \geq h_{12}$, where $h_{12} = \dfrac{1+2\sqrt{y}}{4-\sqrt{y}}$.

In order to verify condition (iii) we have to carry out the same analysis for firm 2:

Given p_1^{N1}, the profit function of firm 2, $B_2(p_1^{N1}, p_2)$ has only one maximum for $p_2 \in [o,+\infty)$, that is reached at $p_2^{**} = p_2^{N1}$, thus we have $B_2(p_1^{N1}, p_2^{N1}) \geq B_2(p_1^{N1}, p_2) \forall p_2$.

Therefore, combining conditions i, ii, iii, we obtain that (p_1^{N1}, p_2^{N1}) is a Nash price equilibrium for $h \geq h_{11}$ and $h \geq h_{12}$.

However, we also verify that $h_{12} > h_{11} > y$, (See figure 3), therefore (p_1^{N1}, p_2^{N1}) is a Nash price equilibrium in the region R_1 defined as:

$R_1 = \{(h, y) | \; h \geq h_{12}\}$ (See figure 4)

Proof of Proposition 3:

In order for (p_1^{N2}, p_2^{N2}) to be a price equilibrium, the following conditions must be met:

(i') $p_1^{N2} - p_2^{N2} \in I_3^1$

(ii') $B_1(p_1^{N2}, p_2^{N2}) \geq B_1(p_1, p_2^{N2}) \forall p_1$

(iii') $B_2(p_1^{N2}, p_2^{N2}) \geq B_2(p_1^{N2}, p_2) \forall p_2$

Condition (i') implies that

a) $y(2h - 1 - y) \leq p_1^{N2} - p_2^{N2}$ Which holds if and only if, $h \leq h_{21}$ where $h_{21} = \dfrac{y(5+y)}{2(2y+1)}$

b) $p_1^{N2} - p_2^{N2} \leq y(1-y)$ Which is always true.

So (p_1^{N2}, p_2^{N2}) could be a Nash price equilibrium if y and h belong to the set:

$R_{21} = \{(h, y) | \; h \geq h_{21}\}$

In order to verify condition (ii'),

Given p_2^{N2}, and considering R_{21}, we verify that $B_1(p_1, p_2^{N2})$ admits a global maximum in $p_1 = p_1^{N2}$.

condition (iii'),

Similarly, given p_1^{N2}, and considering R_{21}, we verify that $B_2(p_1^{N2}, p_2)$ admits a global maximum in $p_2 = p_2^{N2}$.

Finally, (p_1^{N2}, p_2^{N2}) is a Nash price equilibrium in region R_2 defined as:

$R_2 = \{(h, y) | y \leq h \leq h_{21}\}$ (See figure 5)

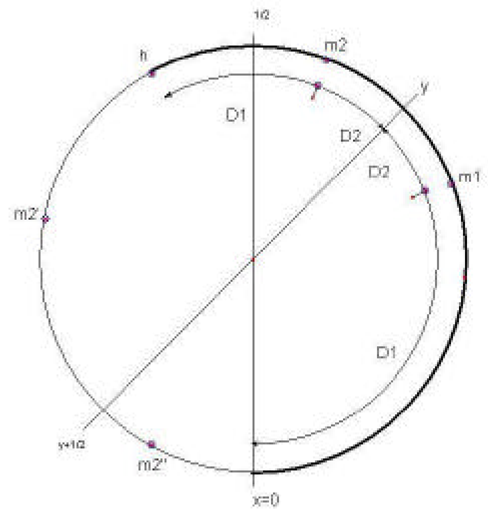

Figure 1

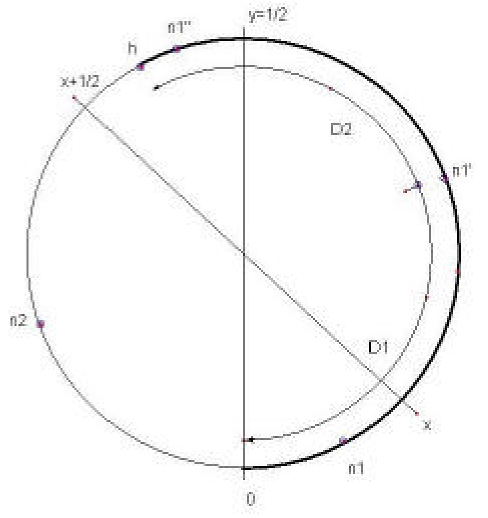

Figure 2

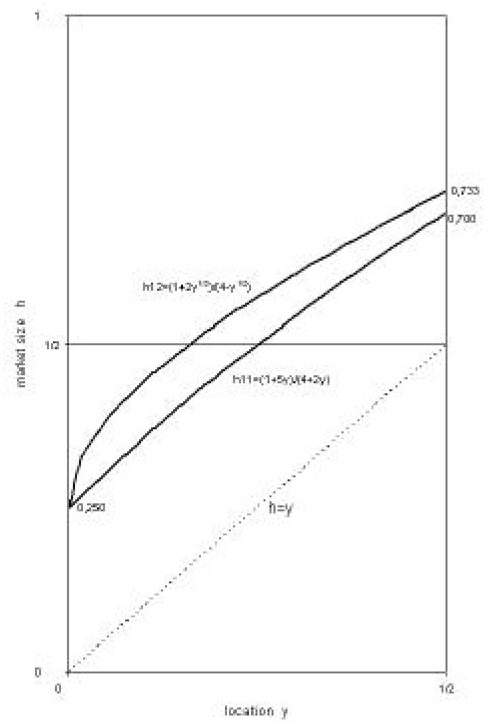

Figure 3

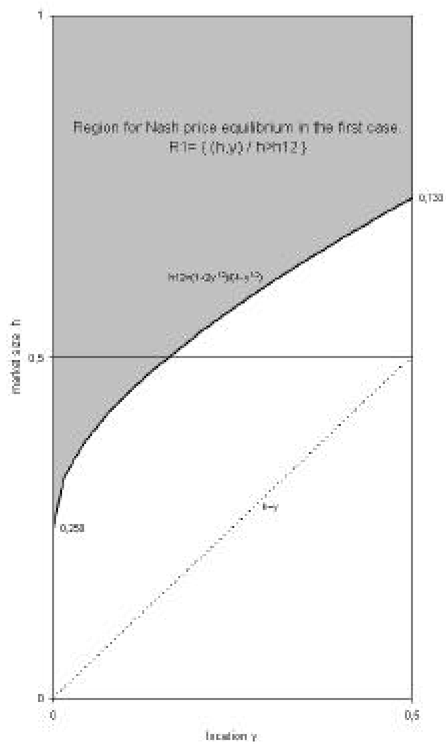

Figure 4

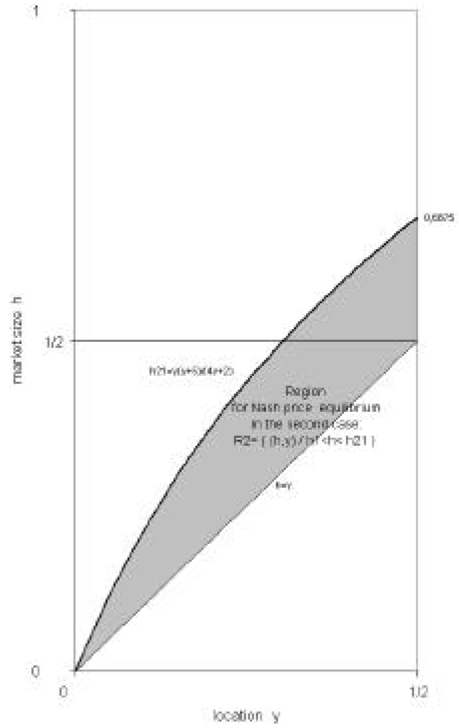

Figure 5

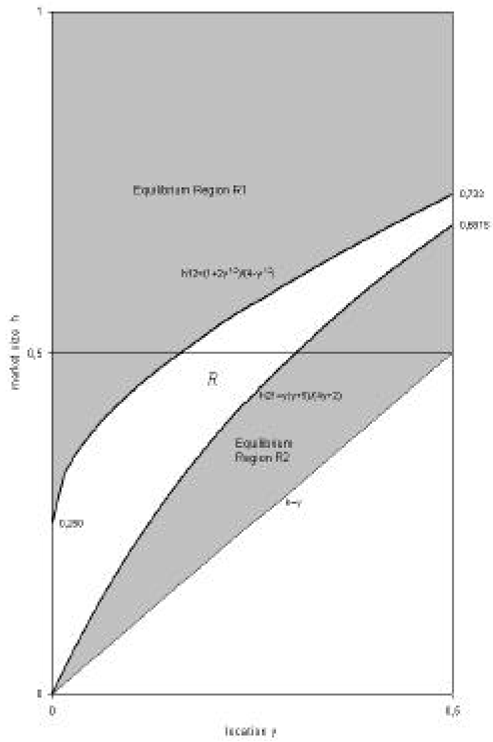

Figure 6

6.- References

Anderson, S. P., 1986. Equilibrium existence in the circle model of product differentiation. London Papers in Regional Sciences Series 16, 19-29.

Anderson, S. P., 1988. Equilibrium existence in the linear model of spatial competition. Economica 55, 479-491.

D'Aspremont C., Gabszewicz J. J., Thisse J. F., 1979. On Hotelling's stability in competition. Econometrica 47, 1145-1150.

De Frutos M. A., Hamoudi H., Xarque X., 1999. Equilibrium existence in the circle model with linear quadratic transport costs. Regional Sciences and Urban Economics 29(5), 605-615.

De Frutos M. A., Hamoudi H., Xarque X., 2002. Spatial competition with concave transport costs. Regional Sciences and Urban Economics 32, 531-540.

Economides, N., 1986. Minimal and maximal product differentiation in Hotelling's duopoly. Economic Letters 21, 67-71.

Gabszewicz J. J., Thisse J. F., 1986. *Location Theory*. Harwood Academic Publishers, Switzerland.

Gabszewicz J. J., Thisse J. F., 1986. On the nature of competition with differentiated products. The Economic Journal 96, 160-172.

Hamoudi H., Moral M. J., 2005. Equilibrium existence in the linear model: concave versus convex transportation costs. Papers in Regional Sciences 84, 201-219.

Hotelling, H., 1929 Stability in Competition. The Economic Journal 39, 41-57.

Salop S. C., 1979. Monopolistic competition with outside goods. Bell Journal of Economics 10, 141-156.