Direction of technical change, endogenous fertility, and patterns of growth

Job market paper

Mehdi Senouci*

October 12, 2012

Abstract

What type of technical progress is able to increase income per capita, instead of merely translating into higher fertility? To investigate this question, this paper first sets up an OLG growth model with capital, land and endogenous fertility. At each date, children compete with physical capital as a means of saving for the young. This framework is then put into motion by continuous *neutral* and *investment-specific* technical change. Neutral technical change leads to well-known Malthusian dynamics and cannot make the wage rate grow asymptotically. On the contrary, investment technology alters the relative price of capital and children and so also affects the households' accumulation/fertility decisions. If capital and labor are strict substitutes in the production function, continuous investment-specific technical change results in long-term growth of per capita income. When the direction of technical change is made endogenous, the agents most often tend to undertake R&D that increases neutral productivity, leading to stagnation of per capita income. The theory is used to interpret some evidence on the first steps of the Industrial Revolution.

*Paris School of Economics. 48, boulevard Jourdan, 75014 Paris, France. (+33)1 43 13 63 14, senouci@pse.ens.fr.
I am most grateful to Daniel Cohen for his careful ideas and advices, as well as to Oded Galor, Basile Grassi, Mathias Lé, Andrea Matranga, Bertrand Wigniolle and participants at PSE seminars for comments and suggestions. I wish to thank Robert Allen for sharing and letting me use his estimates of factor shares.

Introduction

In his 1798 *Essay on the principle of population*, Malthus first enunciated his fundamental law on the mutual regulation of wages and population. In the standard Malthusian model, decreasing marginal returns to labor and land in fixed quantity ensure that any increase in population results in lower wages and lower fertility, so that population and income tend to return to their initial levels. A consequence of that view is that technical progress cannot improve living standards: since technology has the adverse consequence of increasing fertility, an increase in productivity ultimately results in a denser, but not richer, population.

In recent years, an important strand of research has established considerable empirical support for this law for the pre-industrial world[1]. However, history proved that England was already experiencing its growth takeoff when Malthus was preaching restraint in individual fertility[2]. In the eighteenth century, aggregate income started to grow faster than population. By today, the simultaneous occurence of sustained growth and low fertility in industrialized countries is evidence that the Malthusian spiral has completely vanished there.

To account theoretically for the different regimes of growth within one model is of great interest for our understanding of history as well as of the present. The most decisive step in this way is clearly attributable to the literature on human capital and the quality vs. quantity of children. Oded Galor and other advocates of the 'Unified Growth Theory'[3] have built an impressive coherent structure aiming at describing all growth regimes, and deciphering the links between them. The first key assumption is that slow technical change is fertility-prone whereas rapid technical change is human capital-prone; the second one is that higher populations experience higher productivity growth[4]. Population growth is then the necessary prerequesite of the passage to the 'Post-Malthusian' regime, which is inevitable in the long-term.

One can feel uncomfortable with this picture of history. For centuries, China has produced decisive advances in science and technics, but this progress led to a very numerous population[5] and no growth of income per capita, until the last decades. In constrast, England never had to meet the Chinese density standard to experience its Industrial Revolution and its growth takeoff. It seems rather that various paths are possible.

[1] See in particular Pomeranz (2000), Clark (2007) and Ashraf and Galor (2011).

[2] "[Man's] duty is intelligible to the humblest capacity. It is merely, that he is not to bring beings into the world, for whom he cannot find the means of support. [...] If he cannot support his children, they must starve" (Malthus (1820)).

[3] See Galor and Weil (1999, 2000), Galor (2005, 2011) and Ashraf and Galor (2011). See also Jones (2001), Hansen and Prescott (2002), Lucas (2002), Weisdorf (2004, 2008) and Lagerlof (2006).

[4] Kremer (1993).

[5] According to Angus Maddison's data, by 1 CE China already made up one-fourth of world population.

I present a model reflecting the view that technological change can lead to different outcomes according to its *direction*. I distangle the effects of *neutral* technical change – i.e. technical change that lowers the supply cost of *all* goods – and of *investment-specific* technical change – which decreases the supply cost of physical capital exclusively. I prove that the first type of technical change is always Malthusian, while the second type can bring a sustained increase in income per capita.

The model is one of standard overlapping generations, with labor, capital and land. Fertility is endogenous and children are an asset: parents simply extract some fraction of their children' wage income. So children compete with capital as a means of saving for the young, like in Raut and Srinivasan (1994).

The crucial parameter is the technical elasticity of substitution between capital and labor inputs in the production function, which guides the terms of the arbitrage between these two assets. I show that when capital and labor are substitutes in the production function, increasing investment productivity leads to a re-allocation of the youngs' portfolio away from children and towards capital, which eventually permits to overcome the Malthusian constraint. When the direction of technical change is made endogenous, I prove that investment-specific R&D must be exceptionally productive to be undertaken, so that most often people choose to improve on neutral technology only, leading to an endogenous stagnation of living standards.

The model intends to be one of the prime steps of modern growth, say from around 1780 to 1850 in England. In doing so, I present a 'capital rather than children' story of the demographic transition, rather than a 'children quality rather than quantity' and propose some evidence in support of it[6]. The model attempts to illustrate why technical progress must be biased towards investment for growth to happen, and to discuss how investment-specific technical change, rather than human capital accumulation, is likely to have been the prime engine of the English growth takeoff.

The key to the results of the model lies in the intergenerational externalities inherent to the OLG structure of the model. Neutral technical change increases fertility, because the agents do not internalize the effect of their fertility on the well-being of future generations.

In the contrary, investment-specific technical change might make fertility *decrease* and so bring a positive intergenerational externality. In the case capital and labor are strict substitutes in the production function, continuous investment technical change

[6] To my knowledge, the only attempt to link growth takeoff with investment technology is Fernández-Villaverde (2001), who takes a quite different approach than the one in this paper.

then translates in long-run growth of income per capita. When capital and labor are complements, investment technology is, like neutral technology, Malthusian in the long term.

The theory presented suggests that growth of income per capita is nothing but 'inevitable' (Galor 2005, p. 173) but rather is the consequence of some specific direction of the evolution of technology, coupled with some specific form of the production function – namely, that capital and labor are strict substitutes.

The rest of the paper is organized as follows. Section 1 sets up the structure of the model. Section 2 introduces steady neutral and investment-specific technological change and analyzes the resulting patterns of growth. Section 3 presents two extensions of the model. Section 4 provides some consistent evidence on the Industrial Revolution in England. Section 5 concludes.

1 The Malthusian framework with capital and land

1.1 The structure of the model

The economy is made of overlapping generations of identical people living for two periods. There are two goods in the economy, one consumption good and one capital good.

All agents share the same utility function over consumption at both periods of life:

$$U_t = u\left(c_t^y, c_{t+1}^o\right) = \left(c_t^y\right)^{1-s}\left(c_{t+1}^o\right)^s \tag{1}$$

where $s \in (0,1)$.

People are endowed with one unit of labor when young, for which they earn the competitive wage rate w_t (unless noted, all quantities are in terms of the consumption good). To consume when old, they must save. There are two assets in the economy: capital and children.

Capital is produced competitively from the output via a linear technology. At each date t, one unit of output can be turned into q_t units of capital. q_t is an exogenous time-varying parameter reflecting *investment-specific* productivity. Each unit of capital then brings its holder the rate of profit r_{t+1} and depreciates completely at the period after[7]. Initial capital level is given at K_0 and is equally distributed between the old agents at date $t = 0$.

[7]Notice that our assumptions ensure that the agents' rational forward-looking calculus at t stops at $t+1$, making the model readily tractable.

Each child costs ρ units of the consumption good, and surrenders a fraction $\lambda \in (0,1)$ of his wage income w_{t+1} to his parent at the next period.

The two assets are perfect substitutes: the young simply choose the asset with the highest rate of return and, in case both are equal, hold any of the two assets indifferently. In doing so, the young perfectly anticipate next period's wage and profit rates, and so the capital accumulation/fertility choices of other people of the same cohort.

Production uses three inputs: labor, capital and land. We denote by L_t the young population at date t.[8] Capital input at date t, K_t, is equal to the aggregate stock accumulated by the generation born at date $t-1$. Land quantity is fixed at X.

Output at date $t \geq 0$ is given by the following constant returns to scale production function:

$$Y_t = A_t F(K_t, L_t, X) = A_t \left\{ \left(aK_t^{\frac{\sigma-1}{\sigma}} + (1-a)L_t^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \right\}^{1-b} X^b \tag{2}$$

with $0 < a < 1$, $0 < b < 1$ and where A_t is an exogenous time-varying parameter reflecting *neutral* productivity level and $\sigma > 0$ denotes the (fixed) elasticity of substitution between capital and labor. To ensure that marginal returns are decreasing, we assume that $\sigma < \frac{1}{b}$ in the whole paper. If $\sigma \in (0,1)$ then capital and labor are complements, if $\sigma > 1$ they are substitutes[9]. If σ is exactly equal to one, then production takes the following Cobb-Douglas form:

$$Y_t = A_t K_t^{a(1-b)} L_t^{(1-a)(1-b)} X^b \tag{3}$$

There are no property rights over land, so that the land share of income simply vanishes at each period. Similar assumptions are made, among others, by Galor and Weil (2000) and Weisdorf (2008). Here, because the production function is of Cobb-Douglas type in land and the capital/labor aggregate, land share is a constant fraction of gross production, so the assumption that nobody can own land does not have a crucial effect on the patterns of growth and stagnation.

Perfect competition prevails in the markets for capital and labor, and so factor prices are equal to their respective marginal products. Hence, for all $t \geq 0$:

$$w_t = \frac{\partial Y_t}{\partial L_t} = A_t(1-a)(1-b) \left(aK_t^{\frac{\sigma-1}{\sigma}} + (1-a)L_t^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1-b\sigma}{\sigma-1}} L_t^{-\frac{1}{\sigma}} X^b$$

[8] L_0 and L_{-1}, respectively the young and old population at date $t = 0$, are given.

[9] See Arrow, Chenery, Minhas and Solow (1961) and de La Grandville (2009).

$$r_t = \frac{\partial Y_t}{\partial K_t} = A_t a(1-b)\left(aK_t^{\frac{\sigma-1}{\sigma}} + (1-a)L_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1-b\sigma}{\sigma-1}} K_t^{-\frac{1}{\sigma}} X^b$$

Because of constant returns to scale, we can write those two quantities in intensive terms:

$$w_t = A_t(1-a)(1-b)\left(ak_t^{\frac{\sigma-1}{\sigma}} + 1 - a\right)^{\frac{1-b\sigma}{\sigma-1}} x_t^b \quad (4)$$

$$r_t = A_t a(1-b)\left(ak_t^{\frac{\sigma-1}{\sigma}} + 1 - a\right)^{\frac{1-b\sigma}{\sigma-1}} k_t^{-\frac{1}{\sigma}} x_t^b \quad (5)$$

where $k_t = \frac{K_t}{L_t}$ and $x_t = \frac{X}{L_t}$ respectively denote capital per worker and land per worker ratios. k_0 and x_0 are given by history.

1.2 The dynamical system

1.2.1 The intertemporal equilibrium

In this part, we rigorously derive the evolution of the system for any sequences of technology $(A_0, A_1, ...)$ and $(q_0, q_1, ...)$.

The simplest point to deal with is the evolution of the land-labor ratio. If each young of cohort t chooses to have $1 + n_t$ children (with $n_t > -1$), then population evolves according to:

$$\forall t \geq 0, \; L_{t+1} = (1 + n_t)L_t \quad (6)$$

From the definition of x and equation (6) it comes immediately that:

$$\forall t \geq 0, \; x_{t+1} = \frac{x_t}{1 + n_t} \quad (7)$$

The next point is to analyze the dynamics of the capital-labor ratio. If the young of the date-t cohort choose each to purchase k'_t units of capital and to breed $1 + n_t$ children, then the capital-labor at next period will be at:

$$k_{t+1} = \frac{k'_t}{1 + n_t} \quad (8)$$

So, the capital-labor ratio at date $t + 1$ is determined by the ratio of real capital and children assets in the 'portfolio' of the young at date t. Thus, the evolution of the capital-labor ratio is guided by the evolution of the terms of the arbitrage between capital and children as assets. Real interest rate on children is:

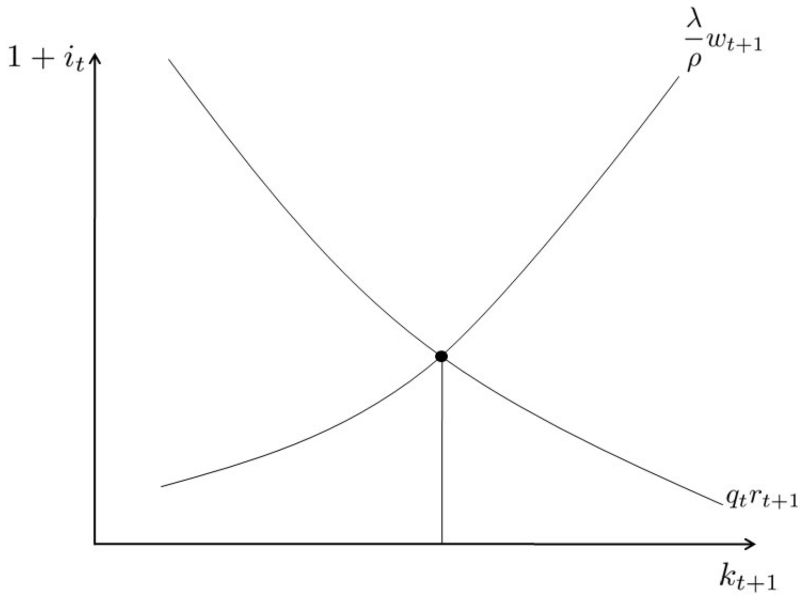

Figure 1: *Equalization of interest rates between dates t and $t+1$ on children and capital assets, and equilibrium period $t+1$ equilibrium capital-labor ratio.*

$$1 + i^C_{t+1} = \frac{\lambda w_{t+1}}{\rho} \tag{9}$$

From equation (4), $1 + i^C_{t+1}$ is an increasing function of expected period $t + 1$ capital-labor ratio. Real interest rate on capital is:

$$1 + i^K_{t+1} = q_t r_{t+1} \tag{10}$$

and is decreasing in k_{t+1}. The situation is depicted in figure 1. At equilibrium the two interest rates must be equal: otherwise people would all choose to hold the same asset, making the other one infinitely more attractive. For instance, if one expects all the others to save in the form of children and hold no capital, then next period profit rate will be infinite, making capital a better bargain. Consequently, at equilibrium, the young have some children and purchase some capital.

From first-order conditions (4) and (5), the two interest rates are:

$$1 + i^C_{t+1} = \frac{\lambda}{\rho} A_{t+1}(1-a)(1-b)\left(a k_{t+1}^{\frac{\sigma-1}{\sigma}} + 1 - a\right)^{\frac{1-b\sigma}{\sigma-1}} x^b_{t+1} \tag{11}$$

$$1 + i^K_{t+1} = q_t A_{t+1} a(1-b)\left(a k_{t+1}^{\frac{\sigma-1}{\sigma}} + 1 - a\right)^{\frac{1-b\sigma}{\sigma-1}} k_{t+1}^{-\frac{1}{\sigma}} x^b_{t+1} \tag{12}$$

Equalization of both interest rates then yields the very simple relationship:

$$k_{t+1} = \chi q_t^{\sigma} \tag{13}$$

where $\chi \equiv \left(\frac{\rho}{\lambda}\frac{a}{1-a}\right)^{\sigma}$.[10]

Equation (13) determines the capital-labor ratio for each period $t \geq 1$ as a function of investment-specific productivity at the period before. It can thus can be viewed as the golden rule of the model. Two remarks are worth making: first, capital-labor ratio is independent of A_t. This property holds because neutral technology does not influence the tradeoff between capital and children for savers, since it increases the two interest rates proportionally[11], nor the relative supply cost of consumption and investment goods. Secondly, in the absence of any change in q, the capital-labor ratio reaches its steady-state value at $t = 1$. Capital accumulation is not an issue in this economy, since fertility is another degree of freedom that leads to the immediate adjustment of the capital-labor ratio. This result is common in models where children are a capital good[12].

Young people of cohort t take their disposable income $(1-\lambda)w_t$ and the interest rate $1 + i_{t+1}$ between both periods of their lives as given and solve the following problem:

$$\left| \begin{array}{ll} \max & \left(c_t^y\right)^{1-s}\left(c_{t+1}^o\right)^{s} \\ & c_t^y + \frac{c_{t+1}^o}{1+i_{t+1}} \leq (1-\lambda)w_t \end{array} \right. \tag{14}$$

Such Cobb-Douglas utility entails a constant saving rate, here exactly equal to s:

$$c_t^y = (1-s)(1-\lambda)w_t$$

$$c_{t+1}^o = (1+i_{t+1})s(1-\lambda)w_t$$

Equilibrium of demand for savings by the young and the supply of children and capital translates into:

$$\forall t \geq 0,\ (1-\lambda)sw_t = \rho(1+n_t) + \frac{k_t'}{q_t}$$

From equations (8) and (13), this relationship can be written as:

[10]Remark that, interestingly, the value of χ does not depend on b, the parameter of land in production. This useful property of the 'neutrality of land' in the equations of arbitrage comes from the assumption that the production function is Cobb-Douglas in land and the capital/labor aggregate.

[11]In figure (1.2.1), an increase in A_{t+1} shifts the two curves up by the same factor, leaving equilibrium capital-labor ratio unchanged.

[12]See Raut and Srinivasan (1994).

$$(1-\lambda)sw_t = (1+n_t)\left(\rho + \frac{k_{t+1}}{q_t}\right)$$

So that:

$$\forall t \geq 0,\ (1-\lambda)sw_t = (1+n_t)\left(\rho + \chi q_t^{\sigma-1}\right) \tag{15}$$

From equations (13) and (15), for any sequences $(A_t)_{t\geq 0}$ and $(q_t)_{t\geq 0}$ and for any initial conditions, the path taken by the economy is determined. The economy evolves according to the following system:

$$\forall t \geq 0,\ \left\{\begin{array}{rcl} w_t & = & A_t(1-a)(1-b)\left(ak_t^{\frac{\sigma-1}{\sigma}} + 1 - a\right)^{\frac{1-b\sigma}{\sigma-1}} x_t^b \\ k_{t+1} & = & \chi q_t^\sigma \\ 1+i_{t+1} & = & \frac{\lambda}{\rho} w_{t+1} \\ 1+n_t & = & \frac{(1-\lambda)sw_t}{\rho+\chi q_t^{\sigma-1}} \\ x_{t+1} & = & \frac{x_t}{1+n_t} \end{array}\right.,\ (x_0,k_0) \text{ given} \tag{16}$$

1.2.2 Convergence and stability without technical change

The first point is to prove that the economy converges to some particular stable steady state in the absence of technological change. Suppose that $A_0 = A_1 = ... = A$ and $q_0 = q_1 = ... = q$. From (16), if w_t is to be constant then $1+n_t$ must be exactly equal to one. From (4), there exists a unique level of the wage rate that induces each young to have one child exactly. So let's define the *subsistence wage* rate w^* by[13]:

$$w^*(q) = \frac{\rho + \chi q^{\sigma-1}}{(1-\lambda)s} \tag{17}$$

Since q_t is fixed, the capital-labor ratio is fixed at its long term level $k = \chi q^\sigma$ from $t = 1$ on. Then, since A is also fixed, from (16) we get:

$$\forall t \geq 0,\ \frac{w_t}{w^*(q)} = 1 + n_t \tag{18}$$

$$\forall t \geq 1,\ \frac{w_{t+1}}{w_t} = \left(\frac{w_t}{w^*(q)}\right)^{-b}$$

[13] I explain later why subsistence wage changes according to q.

$$\forall t \geq 1,\ \frac{w_{t+1}}{w^*(q)} = \left(\frac{w_t}{w^*(q)}\right)^{1-b} \tag{19}$$

Equation (19) proves that w_t converges to $w^*(q)$ for any initial conditions. From $t = 1$ on, the wage rate converges monotonically. What happens between dates $t = 0$ and $t = 1$ is, for now, of little interest, and depends on whether the initial capital-labor ratio k_0 is higher or lower than the long-term capital-labor ratio.

Convergence from $t = 1$ on reflects the traditional Malthusian feedback, which works through land. When $b > 0$, production requires land which is in fixed amount, so there are decreasing returns to capital and labor together.

Suppose, for example, that the economy starts with the long-term capital-labor ratio $k = \chi q^\sigma$ but with a low population, so that $w_0 = 2w^*(q)$. The young at date $t = 0$ then has a fertility rate $1 + n_0 = 2$. The young generation born at $t = 0$ expects the next young generation to share the same capital-labor ratio and technology, but to have a denser population, and so a lower wage rate. The first generation also expects the terms of the arbitrage between saving under the form of capital or children to remain unaffected, since interest rates on capital and children asset shall decrease by the same amount due to the tightening of the land constraint at next period[14]. Consequently, all generations allocate their savings to the two types of assets in the same proportions, which are also the proportions that prevail at steady state. The wage rate, the interest rate and the fertility rate then all decrease smoothly to their long-term values.

If the initial population is higher than at steady state, then the wage rate, the interest rate and the fertility rate converge from below.

This results we can summarize in the following theorem:

Theorem 1.1. *If technology parameters are fixed from $t = 0$ at A and q, then the system converges monotonically to its unique, stable, steady state from $t = 1$ on.*

$$\begin{aligned} w_t &\overset{\infty}{\longrightarrow} w^*(q) \\ 1 + i_{t+1} &\overset{\infty}{\longrightarrow} \tfrac{\lambda}{\rho} w^*(q) \\ 1 + n_t &\overset{\infty}{\longrightarrow} 1 \end{aligned}$$

Equilibrium population is such that the asymptotic amount of land per worker x^* verifies:

$$A(1-a)(1-b)\left(a\chi^{\frac{\sigma-1}{\sigma}} q^{\sigma-1} + 1 - a\right)^{\frac{1-b\sigma}{\sigma-1}} (x^*)^b = \frac{\rho + \chi q^{\sigma-1}}{(1-\lambda)s}$$

[14]See equations (11) and (12).

1.2.3 Investment technology, the relative capital share, and the Malthusian steady state

Subsistence wage in (17) is independent of neutral productivity, like in the Malthusian textbook model[15]. But it depends on investment productivity whenever $\sigma \neq 1$. Namely, w^* is an increasing function of q if $\sigma > 1$ and a decreasing function of q if $\sigma < 1$. This property, which is the key to several subsequent results, stems from to the well-known distributional effects of the relative supply of inputs when the elasticity of substitution between them is not equal to one.

Let's compare the two stady states with investment-specific productivity standing respectively at q and at $q' > q$. At the second steady-state, people hold more capital in *real* terms – that is, in terms of units of capital – from equation (13). But the relative value of capital good to consumption good is also lower. Whether the *value* – in terms of units of the consumption good – of the capital stock owned by each young increases or decreases at steady state depends on the elasticity of substitution between capital and labor. From (13), at steady state:

$$\frac{k}{q} = \chi q^{\sigma-1}$$

which proves that this value increases if $\sigma > 1$, decreases if $\sigma < 1$ and remains constant if $\sigma = 1$. Since the value of 'children assets' per young is equal to ρ at any steady state, we conclude that savings per young – and so, the wage rate – must be relatively higher (resp. lower) at the second steady state if $\sigma > 1$ (resp. $\sigma < 1$).

This effect can be seen through the perspective of the distribution of income with CES production functions. From the equalization of interest rates in equations (9) and (10) and from equation (13) the macroeconomic ratio of the capital share to the labor share at steady state is:

$$\frac{rk}{w} = \frac{\lambda}{\rho}\frac{k}{q} = \frac{\lambda}{\rho}\chi q^{\sigma-1} \tag{20}$$

If $\sigma > 1$, the production function is such that the relative capital-labor share increases as the capital-labor ratio increases. If $\sigma < 1$, it decreases. If $\sigma = 1$, the production function is Cobb-Douglas and relative shares are constant. But from equation (13), the steady-state capital-labor ratio increases with q.

At steady state, from (17) and (20) the wage rate is linked to the relative shares

[15] See Clark (2007, chapter 2) and Ashraf and Galor (2011) for canonical Malthusian models. See also Pr. ??'s course, available on its website, for a Malthusian model with capital accumulation.

of labor and capital through the equation:

$$w^*(q) = \frac{\rho}{\lambda} \frac{\lambda + \frac{rk}{w}}{(1-\lambda)s}$$

This relationship holds for all values of σ and it simply states that, *ceteris paribus*, steady-state wage increases with the relative capital-labor share. When more income goes to capital, more income goes to the old – who cannot have children.

When $\sigma > 1$, an increase in q yields a second-round distributional effect in favor of capital which discourages fertility and makes the Malthusian constraint on the wage rate less stringent, between the first and second steady state. If $\sigma < 1$, second round effects is in favor of labor as capital is accumulated in response to an increase in investment productivity, and the wage rate decreases from one steady state to another.

Some comparative statics results on the Malthusian steady state are presented in appendix A.1

1.3 The dynamic effects of neutral and investment-specific technology shocks

Before turning to the analysis of the growth path of an economy witnessing steady technological progress, we first investigate the dynamic consequences of technology shocks of both types.

1.3.1 Neutral technology shock

Suppose that the economy is initially on its steady-state described by $\left(w^*(q), \frac{\lambda}{\rho}w^*(q), x^*(A,q)\right)$ and witnesses at $t = \tau$ a sudden and forever shock A, which unexpectedly jumps to $A' > A$. We already know that it will converge to its new steady state $\left(w^*(q), \frac{\lambda}{\rho}w^*(q), x^*(A',q)\right)$. Since subsistence wage is independent of A, we simply denote it by w^* in this part.

Since inputs per worker k_τ and x_τ were determined at period $\tau - 1$, from equation (4) the wage rate immediately increases one-for-one with A'/A. From equation (13), capital-labor ratio is unaffected at each period. This means that, after the shock, the increase in savings is distributed between capital and children in the same proportion than before: in short, the young have more children and purchase more capital, so that each of their children will be endowed with exactly as much capital as their parents were. In the end, neutral technology shocks do not have any effect on the relative prices and quantities of inputs nor on the relative price and quantity of capital good, nor at steady state, nor at any date. Adjustment goes solely through population and can be dubbed as 'purely' Malthusian. Successive generations are richer or poorer only as result of the difference in the amount of land per people between them.

The economy ends up with a lower land-per-worker ratio $x^{*\prime}$ such that:

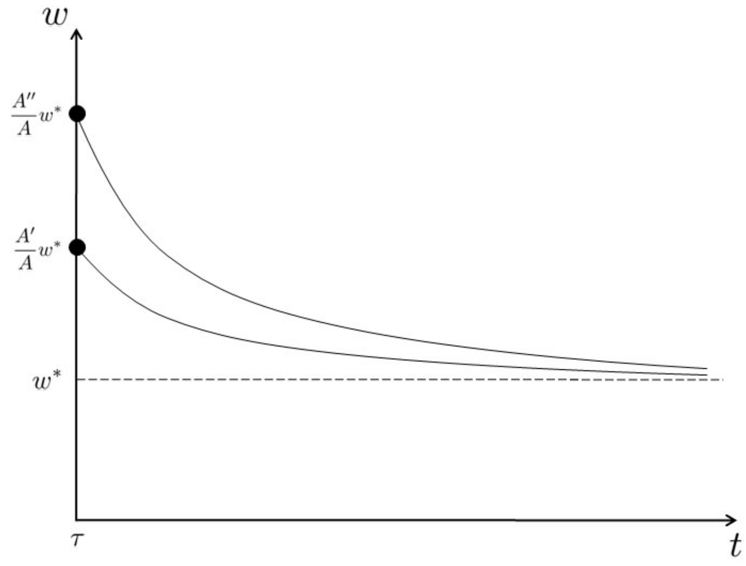

Figure 2: *The dynamic evolution of the wage rate after a shock on neutral technology level from A to $A' > A$ and from A to $A'' < A'$.*

$$x^{*\prime} = \left(\frac{A'}{A}\right)^{-\frac{1}{b}} x^* \tag{21}$$

The post-shock wage rate (and, consequently, interest rate) evolves like:

$$\begin{cases} w_\tau = \frac{A'}{A} w^* \\ \forall t \geq \tau,\ \frac{w_{t+1}}{w^*} = \left(\frac{w_t}{w^*}\right)^{1-b} \end{cases} \tag{22}$$

The fertility rate, here always proportional to the wage rate, converges according to exactly the same pattern: at date $t = \tau$, it jumps from 1 to A'/A. It then evolves like:

$$\forall t \geq \tau,\ 1 + n_{t+1} = (1 + n_t)^{1-b} \tag{23}$$

Convergence of the wage rate is illustrated on figure 2. Remark that nor the elasticity of substitution nor the level of investment-specific technology affects the dynamic path of the wage rate, which only depends on the value of land coefficient b (the more land is important in production, the quicker convergence takes place).

1.3.2 Investment-specific technology shock

Investment-specific technology shocks have richer and more nuanced consequences since they *do* alter relative quantities and prices of inputs and outputs. Suppose now that investment-specific technology jumps at date $t = \tau$ from q to $q' > q$. We already know that monotonic convergence of the wage rate, interest rate and the fertility rate will happen from $t = \tau + 1$ on, still the transition between dates τ and $\tau + 1$, during which the capital-labor ratio is adjusting, is of special interest.

The wage rate does not increase at $t = \tau$ since inputs are fixed from the period before and there are no productivity gains in the sector producing the output: immedately after the shock, $w_\tau = w^*(q)$. Consequently, the amount of savings per young remains constant at $(1 - \lambda)sw^*(q)$, but their division between children and capital changes in a direction depending on the elasticity of substitution[16]. When $\sigma > 1$, young people tend to increase the value of their capital stock, and so they lower their fertility rate below one immediately after the shock. If $\sigma < 1$, the young initially increase its fertility rate above one.

If $\sigma > 1$, period-$\tau + 1$ wage rate is greater than w_τ because capital and land inputs per worker both increase between dates τ and $\tau + 1$. When $\sigma < 1$, agents initially reduce the value of their capital holding and increase their fertility rate above 1, making population increase between dates τ and $\tau + 1$. Then, the capital-labor ratio is higher at $t = \tau + 1$ but the land-labor ratio is lower. Consequently, the wage rate may increase or decrease between τ and $\tau + 1$.

On the path that the economy takes immediately after the shock, we can now state the following proposition:

Proposition 1.1. *Let the economy be initially on its steady state corresponding to a level of investment-specific productivity q. At date $t = \tau$, an investment-specific technological shock makes q increase to $q' > q$ forever. Then, the wage rate remains constant immediately after the shock:*

$$w_\tau = w^*(q) \tag{24}$$

Fertility rate increases above 1 if capital and labor are complements, and decreases below 1 if capital and labor are substitutes. If $\sigma > 1$, the wage rate increases between dates τ and $\tau + 1$. If $\sigma < 1$, it may increase or decrease.

At date $t = \tau + 1$, the new long-term capital-labor ratio is reached, and there is no more technical change. Consequently, the economy witnesses a classical Malthusian mode of convergence towards its new steady-state wage rate $w^*(q')$:

$$\forall t \geq \tau + 1,\ \frac{w_{t+1}}{w^*(q')} = \left(\frac{w_t}{w^*(q')} \right)^{1-b} \tag{25}$$

[16] Remember the discussion in section 1.2.3.

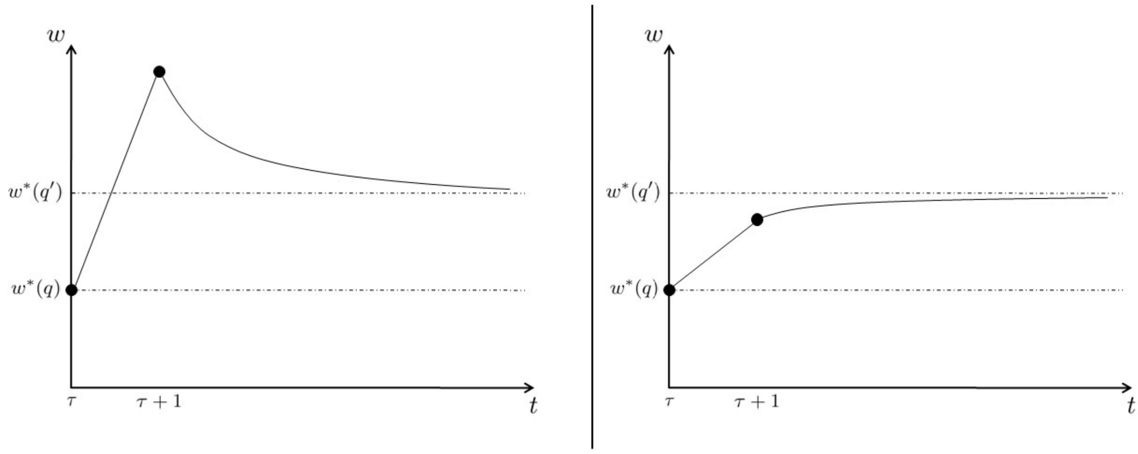

Figure 3: *The dynamic consequences of a positive investment-specific technological shock at* $t = \tau$ *when capital and labor are substitutes.*

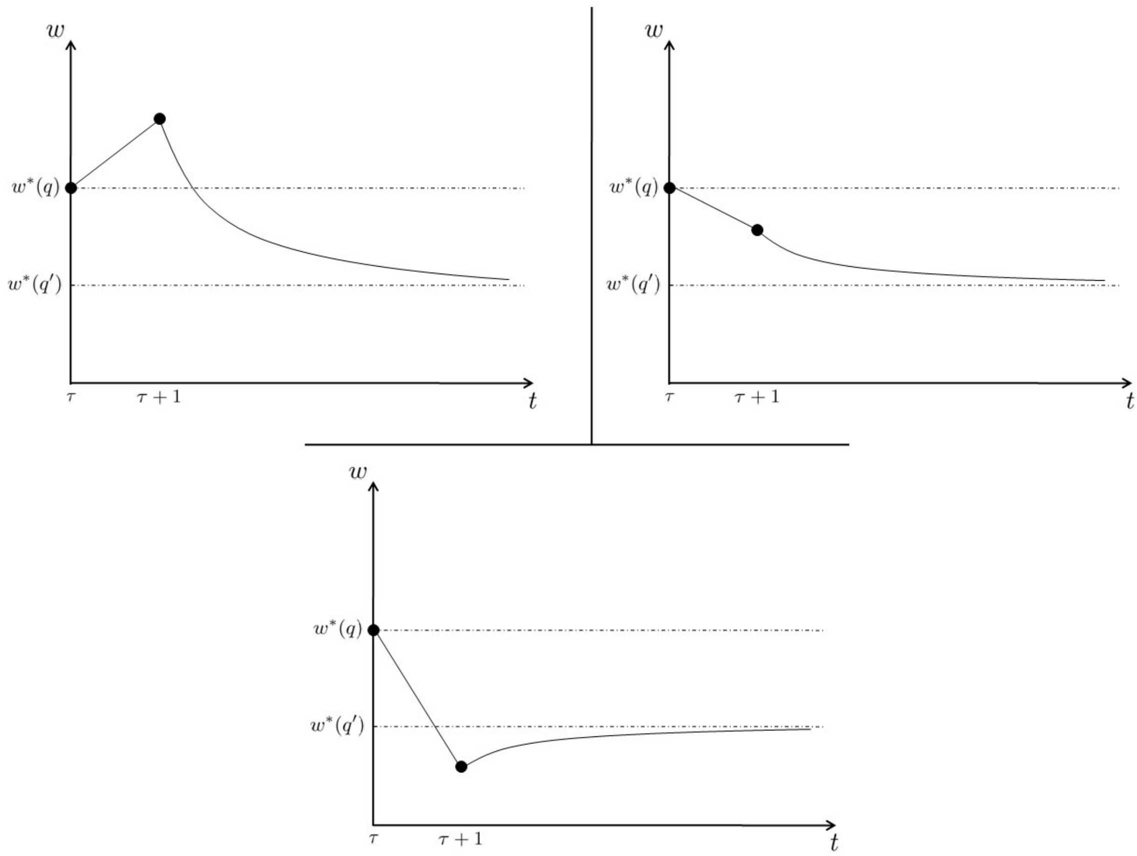

Figure 4: *The dynamic consequences of a positive investment-specific technological shock at* $t = \tau$ *when capital and labor are complements.*

Convergence then takes place from above if $w_{\tau+1} > w^*(q')$ and from below if $w_{\tau+1} < w^*(q')$. In principle, both cases are possible. Possible paths of convergence are described in figures 3 and 4, respectively in the cases where $\sigma > 1$ and $\sigma < 1$.

In some cases, the fertility rate does not follow a monotonic convergence path after an investment-specific shock. For example, in the left case of figure 4, the fertility rate falls below 1 after the shock, but next's period wage increase enough to push it above 1 at $t = \tau + 1$ and during all further transition to the new steady state. Consequently, it is apparently unclear whether population increases or decreases from one steady state to another. Indeed, the new steady-state land-labor ratio verifies:

$$\frac{\rho + \chi q'^{\sigma-1}}{\rho + \chi q^{\sigma-1}} = \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q'^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q^{\sigma-1} + 1 - a} \right)^{\frac{1-b\sigma}{\sigma-1}} \left(\frac{x^{*\prime}}{x^*} \right)^b \tag{26}$$

From equation (26) we can state the following proposition:

Proposition 1.2. *If $\sigma < 1$, then population is higher at the new steady state. If $\sigma > 1$, then population may be higher or lower at the new stady state.*

Proof. The term in $\left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q'^{\sigma-1}+1-a}{a\chi^{\frac{\sigma-1}{\sigma}} q^{\sigma-1}+1-a} \right)^{\frac{1-b\sigma}{\sigma-1}}$ is always strictly greater than one, reflecting the fact that for whatever elasticity of substitution, an investment-specific positive productivity shock increases the steady-state capital-labor ratio and so the wage rate. If σ is strictly lower than 1, the left term in (26) is strictly less than 1, so $\frac{x^{*\prime}}{x^*}$ must be less than one: the land-labor ratio decreases, and population increases between the two steady states. If σ is strictly greater than 1, the left term is also strictly greater than one, so $\frac{x^{*\prime}}{x^*}$ may be above or below one. □

2 Patterns of growth with steady technological progress

We now turn now to the analysis of the dynamics of the economy witnessing continuous technological progress in A or q. People will be now more than experiencing technical change: they will also expect it. We know from section 1.2.1 that the path taken by the economy is determined by (16), with $A_t = (1 + z_A)^t A_0$ if the economy experiences neutral productivity growth, or $q_t = (1 + z_q)^t q_0$ if it experiences investment-specific productivity growth.

2.1 Neutral productivity growth

For simplicity, suppose that the economy is initially on its steady-state without technical progress, for some (A, q). At date $t = 0$, they are informed that some constant, neutral technical change will start at date $t = 1$:

$$\begin{array}{rcl} A_0 & = & A \\ \frac{A_{t+1}}{A_t} & = & 1 + z_A \end{array} \tag{27}$$

Like we saw in the previous section, neutral technical change does not affect the nature of the tradeoff between capital and children assets, and equilibrium capital-labor ratio is not changed at any date. The subsistence wage is constant at w^* and fertility rate at any date verifies that:

$$\forall t \geq 0,\ 1 + n_t = \frac{w_t}{w^*} \tag{28}$$

From (4), the dynamical equation of the wage rate is then:

$$\forall t \geq 0,\ w_{t+1} = (1 + z_A) w_t \frac{1}{(1 + n_t)^b} \tag{29}$$

which can be re-arranged into the following relationship:

$$\frac{w_{t+1}}{w^*} = (1 + z_A) \left(\frac{w_t}{w^*} \right)^{1-b}$$

So, using equation (28), it holds that:

$$\frac{w_{t+1}}{(1 + z_A)^{\frac{1}{b}} w^*} = \left(\frac{w_t}{(1 + z_A)^{\frac{1}{b}} w^*} \right)^{1-b} \tag{30}$$

which proves that the wage rate monotonously converges to $(1 + z_A)^{\frac{1}{b}} w^* > w^*$. The asymptotic fertility rate is then $1 + n^* = (1 + z_A)^{\frac{1}{b}} > 1$.

Neutral technical progress cannot make the wage rate grow asymptotically, since it does not give the young any incentive to decrease their fertility rate. At each date, technical progress makes the wage rate goes up, and the Malthusian 'restoring force' – the term in $(1 + n_t)^{-b}$ – makes the wage rate go down. As w increases, this force gets more and more constraining, since it depends on the cumulated effects of past productivity growth (equation (18)). Inevitably, the wage rate is unable to rise above some level and ultimately stabilizes.

Constant, neutral technological change thus affects the plateau income per capita and the fertility rate, but cannot make income per capita grow asymptotically.

In a nutshell, neutral productivity growth does not induce the agents to store enough of the ensuing increase in savings in the form of capital, rather than children. This, investment productivity might do rapidly enough to counter the Malthusian trap, under the simple condition that capital and labor are (strict) substitutes.

2.2 Investment productivity growth

Suppose now that q is expected to increase steadily from $t=1$ on at the gross rate of $1+z_q$, while A_t remains constant at A. From (13), the capital-labor ratio increases at rate $(1+z_q)^\sigma$ and, from (16), the equilibrium wage rate follows the process:

$$\forall t \geq 1,\ \frac{w_{t+1}}{w_t} = \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a}\right)^{\frac{1-b\sigma}{\sigma-1}} \frac{1}{(1+n_t)^b} \tag{31}$$

This relationship can be written again as:

$$\forall t \geq 1,\ \frac{w_{t+1}}{w_t} = \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a}\right)^{\frac{1-b\sigma}{\sigma-1}} \left(\frac{\rho + \chi q_t^{\sigma-1}}{(1-\lambda)sw_t}\right)^b \tag{32}$$

At the right-hand side, the first term reflects the positive effect of steady capital deepening on the wage rate, while the second term corresponds to the (positive or negative) Malthusian restoring force.

The central result of this part is summarized in the following theorem:

Theorem 2.1. *When investment-specific productivity grows at the constant gross rate of* $1+z_q>1$*:*

- *if* $\sigma > 1$, *the wage rate grows asymptotically at the gross rate of* $(1+z_q)^{\sigma-1} > 1$, *and the fertility rate converges to some constant* $1+n(q) \gtreqless 1$*;*
- *if* $\sigma < 1$, *the wage rate asymptotically converges to its minimum possible steady-state level, which is independent of* σ *and defined by* $\underline{w} = \frac{\rho}{(1-\lambda)s}$. *The fertility rate tends to* 1*;*
- *if* $\sigma = 1$, *the wage rate asymptotically converges to* $(1+z_q)^{a\frac{1-b}{b}} w^* > w^*$ *(where* w^* *denotes the constant subsistence wage rate) and the fertility rate converges to* $(1+z_q)^{a\frac{1-b}{b}} > 1$.

The case where $\sigma > 1$

The method used here (and in section 3.2) is inspired by Jones and Scrimgeour (2008) and their proof of the necessity of pure labor-augmenting technical change.

When $\sigma > 1$, the contribution of capital deepening to the increase of the wage rate at each period is asymptotically constant and positive since:

$$\Gamma_t \equiv \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a}\right)^{\frac{1-b\sigma}{\sigma-1}} \underset{t\infty}{\sim} \left(\frac{q_t}{q_{t-1}}\right)^{1-b\sigma} = (1+z_q)^{1-b\sigma} > 1$$

Let's denote by

$$1 + z_t = \frac{w_{t+1}}{w_t} \tag{33}$$

the gross rate of growth of the wage rate. Dividing equation (32) at dates $t+1$ and t and injecting the definition of $1+z_t$ again leads to an asymptotic equation of evolution for z_t:

$$\frac{1+z_{t+1}}{1+z_t} = \frac{\Gamma_{t+1}}{\Gamma_t}\frac{\left(\frac{\rho+\chi q_{t+1}^{\sigma-1}}{\rho+\chi q_t^{\sigma-1}}\right)^b}{(1+z_t)^b} \underset{t\infty}{\sim} \left((1+z_q)^{\sigma-1}(1+z_t)^{-1}\right)^b \tag{34}$$

Differentiate again relationship (34). Let's define for each date

$$1 + z'_t = \frac{1+z_{t+1}}{1+z_t} \tag{35}$$

Then, by dividing equivalence (34) at dates t and $t+1$ yields:

$$\frac{1+z'_{t+1}}{1+z'_t} \underset{t\infty}{\sim} \left(\frac{1+z_{t+1}}{1+z_t}\right)^{-b} = (1+z'_t)^{-b} \tag{36}$$

And so $z'_t \overset{t\infty}{\longrightarrow} 1$ and $1+z_{t+1} \underset{t\infty}{\sim} 1+z_t$. In equation (34), both terms then tend to 1 and so:

$$1 + z_t \overset{t\infty}{\longrightarrow} (1+z_q)^{\sigma-1}$$

So the wage rate w_t and the interest rate $\frac{\lambda}{\rho}w_{t+1}$ increase steadily along the growth path, and with the same rate than $q_t^{\sigma-1}$. From equation (31) we can conclude that the fertility rate converges to some value $1+n^*$ such that:

$$(1+z_q)^{\sigma-1} = (1+z_q)^{1-b\sigma}(1+n^*)^{-b}$$

And so the fertility rate converges towards $(1+z_q)^{\frac{2-\sigma(1+b)}{b}}$. If $\sigma < \frac{2}{1+b}$, the asymptotic rate of fertility is greater than 1 and so population increases asymptotically. If $\sigma > \frac{2}{1+b}$, population decreases asymptotically. If $\sigma = \frac{2}{1+b}$, then population tends towards some constant[17].

In the case capital and labor are substitutes, the continuous increase in q leads the agents to continually substitute children for capital in their savings portfolio. But as their income increases, they still might be able to push back their fertility rate above 1. The greater the elasticity of substitution σ is, the greater is the growth rate of wage and the more population tends to decrease asymptotically.

[17]Remember that we have supposed from the beginning that σ is always less than $\frac{1}{b}$. Remark that, since $\frac{2}{1+b} < \frac{1}{b}$, the cutoff value of σ at $\frac{2}{1+b}$ is consistent with the model.

The case where $\sigma < 1$

When σ is strictly less than 1, diminishing returns to capital are quick enough to make fertility increase after an investment-specific shock. On the other hand, the contribution of capital deepening vanishes as $t \to \infty$ since:

$$\Gamma_t = \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a} \right)^{\frac{1-b\sigma}{\sigma-1}} \xrightarrow{t\infty} 1.$$

When σ is less than 1, the production function has this unpleasant property that product per worker does not tend to infinity as the capital-labor ratio tends to infinity[18], so that the above result is almost a tautology.

Using equation (32) in the limit then yields:

$$\frac{w_{t+1}}{w_t} \underset{t\infty}{\sim} \left(\frac{\rho}{(1-\lambda)sw_t} \right)^b$$

which immediately proves that the wage rate converges to $\underline{w} = \frac{\rho}{(1-\lambda)s}$, which is independent of z_q. Then, via equation (15) taken at the limit, we conclude that the fertility rate asymptotically converges to exactly 1.

Here, quite amazingly, continual technological progress in the investment sector makes people unambiguously worse off, because it has the second-round effect of increasing labor share. Capital is asymptotically free and in infinite supply, but it induces people to have too many children. Remark that the asymptotic welfare (as described by steady-state wage rate and interest rate) is then simply the lowest possible.

The case where $\sigma = 1$

When the production function is Cobb-Douglas, like already noticed before, investment-specific technical change does not induce a re-balancing of the value of savings between children and capital assets. Then investment-specific technical change is not enough to counter the Malthusian restoring force, and the growth path looks qualitatively much like the one with steady neutral technological progress.

When $\sigma = 1$, equation (32) becomes:

$$\frac{w_{t+1}}{w_t} = \left(\frac{q_t}{q_{t-1}} \right)^{a(1-b)} \left(\frac{\rho + \chi}{(1-\lambda)sw_t} \right)^b = (1 + z_q)^{a(1-b)} \left(\frac{w_t}{w^*} \right)^{-b}$$

from which we conclude that $w_t \xrightarrow{t\to\infty} (1+z_q)^{a\frac{1-b}{b}} w^* > w^*$. The factor $(1+z_q)^{a\frac{1-b}{b}}$, which is strictly greater than 1, is the value of the asymptotic fertility rate.

[18] See de La Grandville (2009), p. 104.

3 Extensions

3.1 The case for more general utility functions

We can now easily drop the assumption that the saving rate of the agents is independent of the interest rate they face. Suppose now that the intertemporal elasticity of substitution is constant, at $\zeta > 0$, so that the agents' preferences are represented by the following CIES utility function:

$$U_t = \left((1-s){c_t^y}^{\frac{\zeta-1}{\zeta}} + s{c_{t+1}^o}^{\frac{\zeta-1}{\zeta}} \right)^{\frac{\zeta}{\zeta-1}} \tag{37}$$

It is well-known that the saving rate s_t then depends only on the expected interest rate $1 + i_{t+1}$ and not on first-period income. Namely, s_t is an increasing function of $1 + i_{t+1}$ if $\zeta > 1$ and a decreasing function of $1 + i_{t+1}$ if $\zeta < 1$. Solving for the young of generation t problem yields:

$$s_t = \frac{\left(\frac{s}{1-s}\right)^{\zeta} (1 + i_{t+1})^{\zeta-1}}{1 + \left(\frac{s}{1-s}\right)^{\zeta} (1 + i_{t+1})^{\zeta-1}} \tag{38}$$

and gross savings per young are at $s_t(1-\lambda)w_t$. Remark that $s_t \xrightarrow{\zeta \to 1} s$. Since at equilibrium the interest rate is at $\frac{\lambda}{\rho} w_{t+1}$, we can write the saving rate as a function of future wage rate:

$$s_t = \frac{\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}{1 + \left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}} \tag{39}$$

The 'golden rule' (13) still holds, since it depends only on the parameters of the production function, and the capital-labor is still fixed at each period at χq_t^{σ} by arbitrage. Then, if savings increase (or decrease) with ζ, the equilibrium proportions of children and capital among those savings will not be altered, so that next-period capital-labor ratio effectively only depends on σ, not on ζ.

Equilibrium equation for savings yields an expression for the rate of fertility at each period $t \geq 0$:

$$1 + n_t = \frac{(1-\lambda)s_t w_t}{\rho + \chi q_t^{\sigma-1}} \tag{40}$$

For any sequences $(A_t)_{t\geq 0}$ and $(q_t)_{t\geq 0}$, the dynamic equation of the wage rate can still be computed the same way than before. For $t \geq 1$:

$$\frac{w_{t+1}}{w_t} = \frac{A_{t+1}}{A_t} \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a} \right)^{\frac{1-b\sigma}{\sigma-1}} \frac{1}{(1+n_t)^b} \tag{41}$$

Combining equations (39), (40) and (41) gives a closed-form representation of the equilibrium path:

$$\frac{w_{t+1}}{w_t} = \frac{A_{t+1}}{A_t} \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a} \right)^{\frac{1-b\sigma}{\sigma-1}} \left(\frac{\rho + \chi q_t^{\sigma-1}}{(1-\lambda) w_t \frac{\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}{1+\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}} \right)^b \tag{42}$$

Since there is now a feedback of next-period wage rate that goes through savings, we must first prove that the intertemporal equilibrium exists and is unique.

Lemma 3.1. *For given $(A_t, A_{t+1}, q_{t-1}, q_t, w_t)$, equation (42) determines a unique equilibrium wage rate w_{t+1}.*

Proof. See appendix A.2. □

In the absence of technical change, subsistence wage $w^*(q)$ verifies the equation:

$$(1-\lambda) w^*(q) \frac{\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w^*(q)\right)^{\zeta-1}}{1 + \left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w^*(q)\right)^{\zeta-1}} = \rho + \chi q^{\sigma-1} \tag{43}$$

and is increasing (resp. decreasing) with respect to q if $\sigma > 1$ (resp. if $\sigma < 1$).

In the absence of technical change, we prove that the system then converges to the subsistence steady state.

Lemma 3.2. *Let A_t and q_t be fixed at A, q. Then the following autonomous system in (w_t):*

$$\frac{w_{t+1}}{w_t} = \left(\frac{\rho + \chi q^{\sigma-1}}{(1-\lambda) w_t \frac{\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}{1+\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}} \right)^b \tag{44}$$

monotonically converges to $w^*(q)$ for any initial w_0.

Proof. See appendix A.3. □

The neutral technology-led path

When q_t is fixed at q while A_t grows at the constant rate $1+z_A$, (42) can be rewritten as:

$$\frac{w_{t+1}}{w_t} = (1+z_A)\left(\frac{\rho+\chi q^{\sigma-1}}{(1-\lambda)w_t\frac{\left(\frac{s}{1-s}\right)^{\zeta}\left(\frac{\lambda}{\rho}w_{t+1}\right)^{\zeta-1}}{1+\left(\frac{s}{1-s}\right)^{\zeta}\left(\frac{\lambda}{\rho}w_{t+1}\right)^{\zeta-1}}}\right)^{b} \tag{45}$$

An increase in the expected wage rate increases the interest rate, which increases savings if $\zeta > 1$ and decreases savings if $\zeta < 1$. In both cases, however, the terms of the capital/children tradeoff are not altered, since q (nor ρ) does not change. Consequently, the capital-labor remains unaffected, and fertility adjusts one-for-one with the variations of savings.

If $\zeta > 1$, an increase in the wage rate has the second-round effect of increasing fertility, and so tend to restore the original state. Remark that the ensuing increase in fertility also makes the wage rate and the profit rate decrease proportionally, like in (4) and (5). In this case, the Malthusian force is reinforced: the saving rate and income are both increasing, both making fertility higher.

When $\zeta < 1$, wealth effect dominates substitution effect and those two forces go in opposite direction: as the wage rate increases, the interest rate (on children) increases and so savings and fertility tend to decrease. So fertility may *a priori* decrease steadily, basically if the saving rate moves more quickly to 0 than the wage rate moves to infinity. We show that it is however impossible when the utility function of the agents pertain to the CIES class.

Take again the definitions of wage growth and acceleration rates z_t and z'_t in equations (33) and (35), then (45) taken at dates t and $t+1$ brings, for any $\zeta > 0$:

$$\begin{aligned} 1+z'_t = \tfrac{1+z_{t+1}}{1+z_t} &= \left(\frac{w_t}{w_{t+1}}\left(\frac{w_{t+1}}{w_{t+2}}\right)^{\zeta-1}\frac{1+\psi w_{t+2}^{\zeta-1}}{1+\psi w_{t+1}^{\zeta-1}}\right)^{b} \\ &= \left(\frac{1}{(1+z_t)(1+z_{t+1})^{\zeta-1}}\frac{1+\psi w_{t+2}^{\zeta-1}}{1+\psi w_{t+1}^{\zeta-1}}\right)^{b} \end{aligned} \tag{46}$$

Suppose that $\zeta > 1$, and that the wage rate tends to infinity. Then asymptotically:

$$\frac{1+\psi w_{t+2}^{\zeta-1}}{1+\psi w_{t+1}^{\zeta-1}} \underset{t\infty}{\sim} \left(\frac{w_{t+2}}{w_{t+1}}\right)^{\zeta-1} = (1+z_{t+1})^{\zeta-1}$$

and so equation (46) yields the following asymptotical equation:

$$\frac{1+z_{t+1}}{1+z_t} \underset{t\infty}{\sim} \frac{1}{(1+z_t)^b} \tag{47}$$

from which it is straightforward to prove that $1 + z_t = \frac{w_{t+1}}{w_t} \xrightarrow{t\infty} 1$. In equation (45), the left-hand side thus tends to 1, and so fertility and the amount of savings per young $(1-\lambda)w_t \frac{\left(\frac{s}{1-s}\right)^\zeta \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}{1+\left(\frac{s}{1-s}\right)^\zeta \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}$ also converge.

Since $w_{t+1} \underset{t\infty}{\sim} w_t$, the following quantity:

$$\frac{w_{t+1}^\zeta}{\left(\frac{1-s}{s}\right)^\zeta \left(\frac{\rho}{\lambda}\right)^{\zeta-1} + w_{t+1}^{\zeta-1}} \tag{48}$$

also converges.

But for all $\zeta > 0$ and for any $\Delta > 0$, function $x \mapsto \frac{x^\zeta}{\Delta + x^{\zeta-1}}$ is strictly increasing[19] from zero to infinity, and so there exists a unique possible steady-state wage rate which is eventually attained, in contradiction with the assumption made that the wage rate diverged to infinity.

Suppose now that $\zeta < 1$, and again that the wage rate tends to infinity. Then:

$$\frac{1 + \psi w_{t+2}^{\zeta-1}}{1 + \psi w_{t+1}^{\zeta-1}} \underset{t\infty}{\sim} 1$$

and (46) asymptotically reduce to:

$$\begin{aligned} & \frac{1+z_{t+1}}{1+z_t} && \underset{t\infty}{\sim} && \frac{1}{\left((1+z_t)(1+z_{t+1})^{\zeta-1}\right)^b} \\ \Longleftrightarrow \quad & \left(\frac{1+z_{t+1}}{1+z_t}\right)^{1-b} && \underset{t\infty}{\sim} && \frac{1}{(1+z_{t+1})^{\zeta b}} \\ \Longleftrightarrow \quad & 1 + z_{t+1} && \underset{t\infty}{\sim} && (1+z_t)^{\frac{1-b}{1-b+\zeta b}} \end{aligned} \tag{49}$$

Since $\frac{1-b}{1-b+\zeta b} \in (0,1)$ for all $\zeta > 0$, (49) proves that $z_t \xrightarrow{t\infty} 0$. Consequently, the left-hand side of (45) tends to 1, and consequently savings per young must also converge. But again $w_{t+1} \underset{t\infty}{\sim} w_t$ and so the quantity:

$$\frac{w_{t+1}^\zeta}{\left(\frac{1-s}{s}\right)^\zeta \left(\frac{\rho}{\lambda}\right)^{\zeta-1} + w_{t+1}^{\zeta-1}}$$

converges, and so the wage rate. Constant intertemporal elasticity of substitution does not induce ‘aggressive’ enough dynamic wealth effects to make the agents reduce their fertility rate so that income per worker can increase asymptotically. We have thus proved the following theorem:

Theorem 3.1. *If the utility function has the CIES form in equation (37) and if the economy witnesses steady neutral productivity growth, then the wage rate, the interest rate and the fertility rate all converge to some constant values as time tends to infinity.*

[19]Derived function is $x \mapsto \frac{\zeta \Delta x^{\zeta-1} + x^{2\zeta-2}}{(\Delta + x^{\zeta-1})^2} > 0$.

The investment-specific technology-led path

Suppose now that q_t grows at the gross rate of $1+z_q$, while neutral technology remains constant at A. Then the wage rate evolves according to:

$$\frac{w_{t+1}}{w_t} = \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a} \right)^{\frac{1-b\sigma}{\sigma-1}} \left(\frac{\rho + \chi q_t^{\sigma-1}}{(1-\lambda) w_t \frac{\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}{1+\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}} \right)^{b} \tag{50}$$

If capital and labor are complements in the production function ($\sigma < 1$), then equation (50), as time tends to infinity, yields:

$$\frac{w_{t+1}}{w_t} \underset{t\infty}{\sim} \left(\frac{\rho}{(1-\lambda) w_t \frac{\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}{1+\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho} w_{t+1}\right)^{\zeta-1}}} \right)^{b} \tag{51}$$

and so the wage rate converges to some stable value in virtue of lemma 3.2.

When capital and labor are substitutes ($\sigma > 1$), we prove the following result:

Proposition 3.1. *The agents have a constant elasticity of intertemporal substitution $\zeta > 0$, and capital and labor are substitutes in the production function ($\sigma > 1$). When investment-specific productivity grows at rate $1+z_q$, the wage rate grows at rate $(1+z_q)^{\frac{\sigma-1}{\zeta}}$ if $\zeta \leq 1$; and at rate $(1+z_q)^{\sigma-1}$ if $\zeta > 1$.*

Proof. See appendix A.4. □

3.2 Endogenizing the direction of technical change

The model has so far been devoted to analyzing the consequences of exogenous technical change on the path taken by the economy. When capital and labor are substitutes, from a long-term perspective the alive agents are unequivocally better off if investment-specific productivity increases, rather than neutral productivity[20]. We are now in a position to tackle the important issue of how the agents *endogenously* choose the direction of technical change. To keep the possibility of growth of income per capita, we suppose in this part that $\sigma > 1$.

The main point here is the conflict of interests between current and future generations. To highlight this side of the problem, let's simply suppose that, at each date, before production, the young collectively choose either to increase neutral technology

[20] On efficiency in models with endogenous fertility, see Conde-Ruiz, Giménez and Pérez-Nievas (2010).

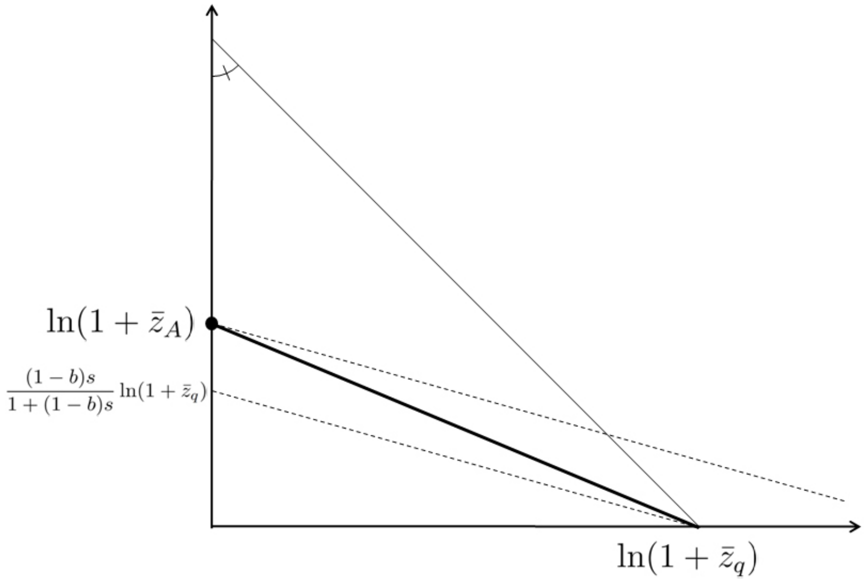

Figure 5: *The log technological expansion frontier (thick line) and the isoquant of indirect utility (in dashed lines) when* $1+\bar{z}_A > \frac{(1-b)s}{1+(1-b)s}$*. The economy undertakes neutral R&D exclusively (•).*

by an amount $1+\bar{z}_A$ or to increase investment productivity by an amount $1+\bar{z}_q$. They could also possibly choose some combination of the two, by increasing A by $(1+\bar{z}_A)^\alpha$ and q by $(1+\bar{z}_q)^{1-\alpha}$, with $\alpha \in (0,1)$ (figure 5). In doing so, they perfectly anticipate present and future consequences of this choice[21], and notably the influence on capital accumulation and fertility. New technology is then made available to the following generation, which then faces the same choice.

When utility function is like described by (1), let's denote by $V(w_t, 1+i_{t+1})$ the indirect utility function of young people at date t with an income of $(1-\lambda)w_t$ and facing an interest rate of $1+i_{t+1}$. Since the young agents save a constant fraction s of their available income at optimum, function V is:

$$V(w_t, 1+i_{t+1}) = ((1-\lambda)(1-s)w_t)^{1-s}\,((1+i_{t+1})(1-\lambda)sw_t)^s$$

$$V(w_t, 1+i_{t+1}) = \Lambda(1+i_{t+1})^s w_t \tag{52}$$

where $\Lambda > 0$ is a constant.

At the beginning of each date t, generation t inherits technological levels A_{t-1} and q_{t-1}, as well as a capital-labor ratio and land-labor ratio are fixed at $k_t = \chi q_{t-1}^\sigma$ and x_t.

[21]Notice that we are abstracting from scale effects.

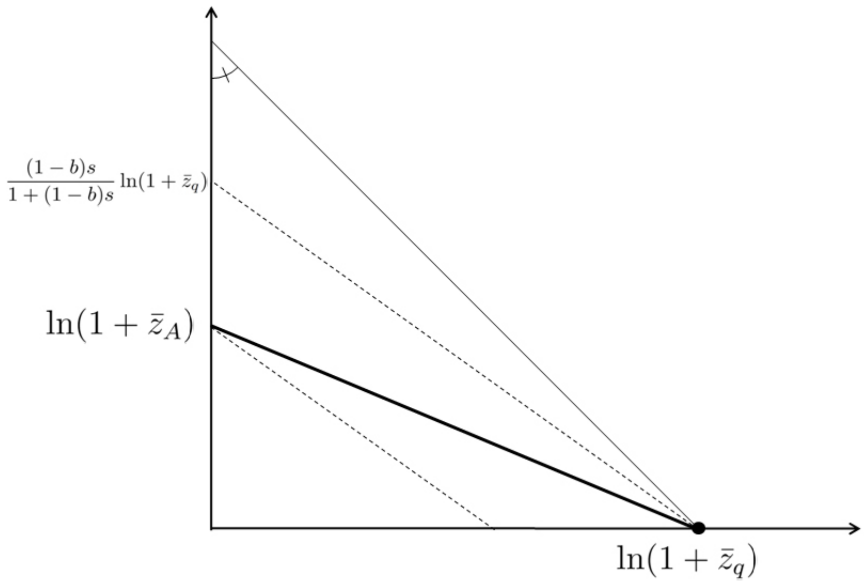

Figure 6: *The log technological expansion frontier (thick line) and the isoquant of indirect utility (in dashed lines) when* $1+\bar{z}_A < \frac{(1-b)s}{1+(1-b)s}$. *The economy undertakes investment-specific R&D exclusively (●).*

If the young agents of date t choose to increase A_t by a factor $1+\bar{z}_A$, then wage rate increases one-for-one as compared as if they did nothing. Fertility also increases one-for-one. Moreover, since in this case q_t remains at q_{t-1}, they expect capital-labor to remain at the inherited level k_{t-1}. From (5), the equilibrium interest rate increases by $(1+\bar{z}_A)^{1-b}$ and so less than one-for-one with A – due to the negative effect of increased fertility.

Indirect utility of the generation born at date t then increases by a factor $(1+\bar{z}_A)^{1+(1-b)s}$ as compared as if no research was undertaken by the young of date t.

If the young chooses to perform investment-specific R&D instead, their wage rate stays the same, and from (15) their fertility decreases by $\frac{\rho+\chi q_{t-1}^{\sigma-1}}{\rho+\chi q_t^{\sigma-1}}$. Equilibrium interest rate increases due to higher capital-labor ratio; in equation (9) the interest rate on children increases by $\left(\frac{a\chi^{\frac{\sigma-1}{\sigma}}q_t^{\sigma-1}+1-a}{a\chi^{\frac{\sigma-1}{\sigma}}q_{t-1}^{\sigma-1}+1-a}\right)^{\frac{1-b\sigma}{\sigma-1}}$. So indirect utility increases by a factor:

$$\left[\left(\frac{a\chi^{\frac{\sigma-1}{\sigma}}q_t^{\sigma-1}+1-a}{a\chi^{\frac{\sigma-1}{\sigma}}q_{t-1}^{\sigma-1}+1-a}\right)^{\frac{1-b\sigma}{\sigma-1}}\left(\frac{\rho+\chi q_t^{\sigma-1}}{\rho+\chi q_{t-1}^{\sigma-1}}\right)^{b}\right]^{s}$$

If q_{t-1} and q_t are high enough, this factor can be approximated by $(1+\bar{z}_q)^{(1-b)s}$. We make this assumption for the rest of this part.

The generation born at t thus will increase neutral technology if:

$$(1+\bar{z}_A)^{1+(1-b)s} > (1+\bar{z}_q)^{(1-b)s}$$

and it will increase investment-specific productivity if:

$$(1+\bar{z}_A)^{1+(1-b)s} < (1+\bar{z}_q)^{(1-b)s}$$

In the case both terms are equal, generation t is indifferent between all locations of the possibility frontier in figure 5. Notice that these constraints do not involve any path-dependence nor stragetic aspect among generations: when the approximation that q is sufficiently high holds, all generations have the same dominant strategy, except when $(1+\bar{z}_A)^{1+(1-b)s} = (1+\bar{z}_q)^{(1-b)s}$.

Therefore, we have proven the following proposition:

Proposition 3.2. *The economy endogenously undertakes investment-specific R&D only if:*

$$\frac{\ln(1+\bar{z}_q)}{\ln(1+\bar{z}_A)} \geq 1 + \frac{1}{(1-b)s}$$

This constraint is more likely to be met if b is close to 0 and if s is close to one. But still investment-specific R&D has to be largely more efficient than neutral R&D. If $b \to 0$ and $s \to 1$, the constraint can be written like:

$$\ln(1+\bar{z}_q) \geq 2\ln(1+\bar{z}_A)$$

which is already a very tight constraint. Otherwise, immediatism leads the young to prefer to increase neutral technology rather than investment technology. The right policy to pursue for a planner with infinite horizon is to tax the young (or the old) to subsidize investment-specific R&D, to the point that neutral R&D is completely discouraged.

4 The Industrial Revolution and the British economy, 1760–1850

The 'Unified Growth Theory' has met an important problem when confronted to the data on the early decades of the Industrial Revolution, i.e. up to the mid-nineteenth century. In a nutshell, the consensus among economic historians is that human capital had little importance during that period. Indeed, according to all latest measures, the skill premium – which is a good measure of the rate of return of an investment in human capital – was low and stagnant during these years[22]. For decades after

[22] See Voth (2003), Galor (2005, 2.3.3), Clark (2005), O'Rourke, Rahman and Taylor (2008) and Rahman (2011).

GDP per capita began growing steadily, there has been no sign of a human capital revolution. Galor (2005, p. 198) claims that "In the first phase of the Industrial Revolution (1760–1830), capital accumulation as a fraction of GDP increased significantly whereas literacy rates remained largely unchanged. Skills and literacy requirements were minimal, the state devoted virtually no resources to raise the level of literacy of the masses, and workers developed skills primarily through on-the-job training (...) Consequently, literacy rates did not increase during the period 1750–1830."

Thus the quantity/quality tradeoff is not relevant to explain the prime steps of the British growth takeoff[23]. I argue in this section that the framework presented in the previous sections provides a better description of the macroeconomics of England for that period, though it fails to account properly for the following growth regime (1850–1914) where human capital was more crucial in the growth process.

According to the conventional Crafts-Harley estimates of GDP and its components, the investment rate grew from 6% in 1760 to 11.7% in 1831[24], meaning that nominal investment grew quicker than nominal GDP by 1.0 percentage point per annum between 1760 and 1830. This simple calculus suggests that there must have been some investment-specific technical change during that period[25].

Another stylized fact of the Industrial Revolution is the low – if any – increase in real wages until well into the nineeteenth century, well after GDP per capita started to takeoff[26].

Figure 7 depicts the evolution of the real wage rate for craftsmen and labourers in London between 1740 and 1913. Data comes from Allen (2001). From 1750 to 1800, real wages actually *decreased* by 20 or 25 percent. From 1800 to 1840, the wage rate increased at a modest pace, barely undoing the previous decrease, and remaining within the historical standards. The real wage rate definitely started its upward trend around 1850. As a result, GDP per capita increased quicklier than the wage rate until the mid-nineetenth century.

The prime steps of growth of income per capita thus came with an increase in the

[23]Proponents of the 'Unified Growth Theory' have attempted to reconcile their view with the data. For instance, Galor and Moav (2004) introduce credit constraints in a growth model to explain the transition from physical to human capital as the main engine of growth, and Galor and Moav (2006) provide a political economy model of lobbying for public provision of human capital.

[24]Galor (2005, p. 206) citing data from Crafts (1985, p. 73).

[25]I am not aware of any research attempting to estimate the relative price of investment to consumption goods during the Industrial Revolution. There is, however, ample evidence of a steady decline in relative capital goods price from the mid-nineteenth century until today (Collins and Williamson (2001)) and anecdotal evidence on the early phase of the Industrial Revolution suggests that it owed much to the spinning machine and the steam engine, both of which are primarily producer goods.

[26]According to most economic historians, GDP per capita started to increase steadily around mid-eighteenth century. Maddison's (2001) data dates back the growth of income per capita to several centuries earlier. Proponents of the 'Industrious Revolution' have also argued that income per capita started to grow long before the Industrial Revolution.

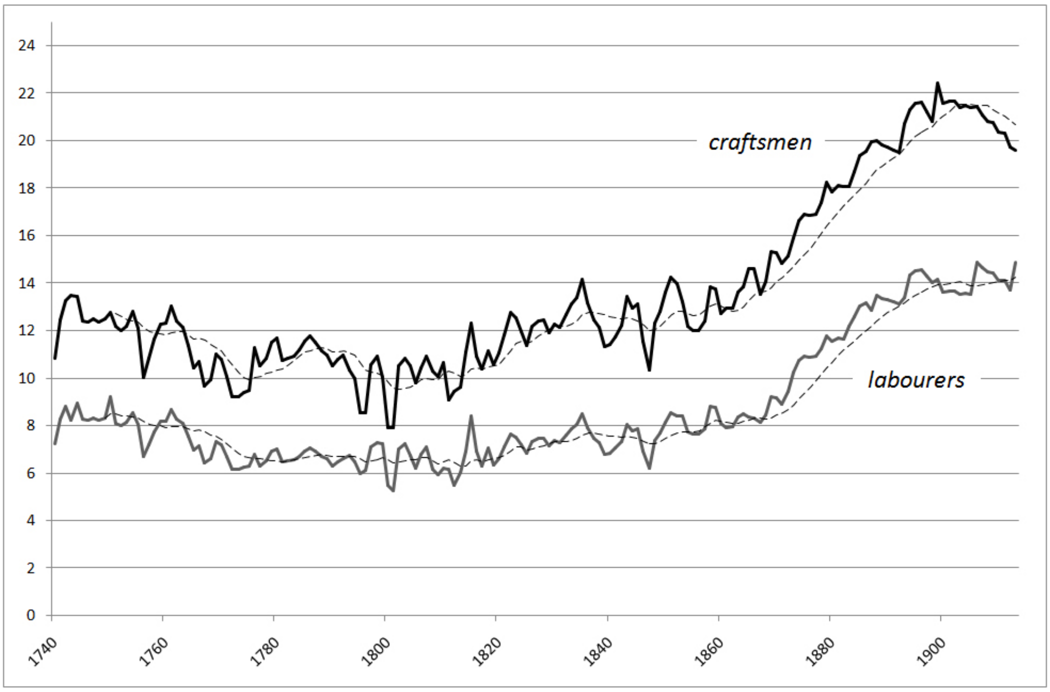

Figure 7: *Real wages of craftsmen and labourers in the construction industry, London, 1740–1913.* Source: *Allen (2001), data available at the author's webpage.*

relative capital share at the expense of relative labor and land shares, a fact that Allen (2009) dubs an "Engel's pause"[27], like illustrated in figure 8. The early fruits of growth went primarily to capitalists, rather than to landowners or laborers, until as late as 1880[28]. This fact can be seen as the dynamic consequence of capital accumulation (due to investment-specific technical change) as the elasticity of substitution between capital and labor is strictly greater than 1.

The model provides some insight on how the escape from the Malthusian trap initially built on physical capital, and why it *had to* reward primarily the capitalists to be possible. If technology parameters are such that the flow of income created by investment-specific technical progress goes primarily to capital, then fertility does not increase enough to completely cancel the incrase in income per person. Capital-labor substitutability is the necessary precondition that makes investment technology efficient in raising income per capita. In the spirit of the model, the deepening of inequality between workers and capitalists was neither the cause nor the consequence

[27]Notably, the decline of the relative labor share came despite substantial increase in labor supply of women and children, as well as in the number of hours worked per year and per male worker. See Voth (2003) and Angeles (2008).

[28]According to Clark's (2009) data, the real wage rate began increasing *more rapidly* than real GDP per capita in the very early nineteenth century. But even in this dataset real GDP per capita began increasing several decades before the real wage rate.

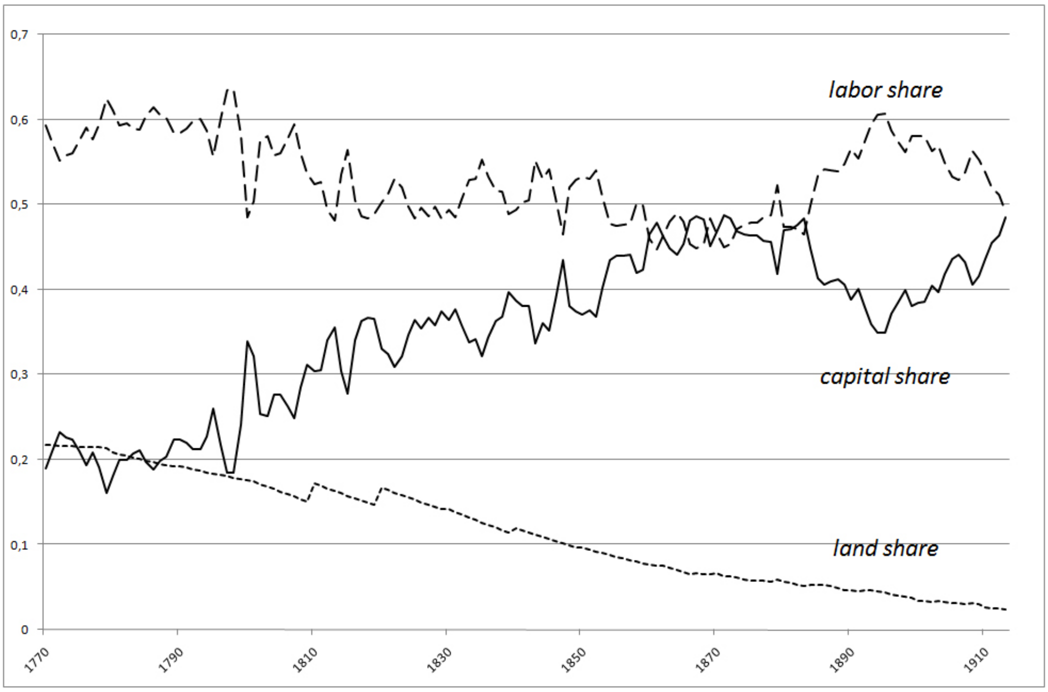

Figure 8: *Allen's (2009) estimates of relative factor shares, 1770–1913.*

of the growth of real income per capita, rather it was capital accumulation – itself due to investment-specific technical change – that drived both.

However, the framework presented in this paper cannot account for the subsequent phase of the Industrial Revolution when the investment rate stabilized (Galor (2005)), when the wage rate grew quicklier than GDP per capita (Allen (2009)), when fertility started to decrease notably (O'Rourke et al. (2008)) and when human capital took the central role in the growth process.

5 Concluding remarks

The consensual evidence on the Industrial Revolution calls for a reconsideration of the capital theory of development. This paper has set up a canonical OLG model where children are a capital good and put it into motion via steady neutral and investment-specific technical change. It came out that investment-specific technical change *only* can lead to a steady increase of individual well-being, at the condition that capital and labor inputs should be strict substitutes in production. The growth path, however, is degenerated: the real wage rate grows asymptotically, but the labor share ultimately tends to zero. This fact is at odds with the second phase of the Industrial Revolution, correctly analyzed by the human capital literature[29], as well as with growth on the very long term, where the labor share seems to be more or less constant.

The literature still misses a theory able to explain the successive stages of growth, and the deep reason for the macroeconomic shift from physical to human capital-centered growth path. A better understanding of the interaction between investment-specific technical change (the capacity to design new, more efficient tools) and human capital (the efficiency in using existing tools) would certainly give us some important new insights into the growth process.

The model also suggests that different economies might experience different qualitative paths, i.e. that not all the economies in the World are on different points of a same path ultimately leading to modern growth. There is no doubt that European bifurcation must have its roots very long back before the Industrial Revolution. Perhaps a better fundamental representation of the growth process would help us to look for the deep roots of the European miracle.

[29] See for instance Becker, Murphy and Tamura (1990) and Becker, Glaeser and Murphy (1999).

A Appendix

A.1 The Malthusian steady state

From the definition of χ in (13), we can re-write steady-state wage and interest rate as:

$$w^*(q) = \frac{\rho + \left(\frac{\rho}{\lambda}\frac{a}{1-a}\right)^{\sigma} q^{\sigma-1}}{(1-\lambda)s} \tag{53}$$

$$\frac{\lambda}{\rho} w^*(q) = \frac{\lambda + \left(\frac{\rho}{\lambda}\right)^{\sigma-1}\left(\frac{a}{1-a}\right)^{\sigma} q^{\sigma-1}}{(1-\lambda)s} \tag{54}$$

And we can derive some comparative statics results:

Proposition A.1.

$$\begin{array}{rcl}
\frac{\partial w^*}{\partial s} & < & 0 \\
\frac{\partial w^*}{\partial a} & > & 0 \\
\frac{\partial w^*}{\partial \rho} & > & 0 \\
w^* & \xrightarrow{\rho\to\infty} & \infty \\
\frac{\partial \frac{\lambda}{\rho} w^*}{\partial \rho} & > & 0 \quad \textit{if } \sigma > 1 \\
\frac{\partial \frac{\lambda}{\rho} w^*}{\partial \rho} & < & 0 \quad \textit{if } \sigma < 1 \\
\frac{\partial (1-\lambda) w^*}{\partial \lambda} & < & 0 \\
w^* & \xrightarrow{\lambda\to 0} & \infty \\
w^* & \xrightarrow{\lambda\to 1} & \infty \\
(1-\lambda)w^* & \xrightarrow{\lambda\to 0} & \infty \\
(1-\lambda)w^* & \xrightarrow{\lambda\to 1} & \varphi > 0 \quad \left(= \frac{1+\rho^{\sigma-1}\left(\frac{a}{1-a}\right)^{\sigma} q^{\sigma-1}}{s}\right) \\
\frac{\partial \frac{\lambda}{\rho} w^*}{\partial \lambda} & > & 0 \quad \textit{if } \sigma < 1 \\
\frac{\partial \frac{\lambda}{\rho} w^*}{\partial \lambda} & \xrightarrow{\lambda\to 0} & 0 \quad \textit{if } \sigma < 1
\end{array}$$

Proof. Straightforward from equations (53) and (54). □

First, everything else equal, an increase in the propensity to save s makes the agents more keen to have children at every level of income, since children are a means of savings. If people chose to put all saving in excess in the form of capital, marginal return of capital would fall while the marginal return of labor – and so, interest rate on children – would rise, in contradiction of the arbitrage reflected in equation (13). Since fertility per young must be exactly one at steady state, an economy with a higher propensity to save ends up with a lower steady-state wage rate. Since the interest rate at steady state is equal to $\frac{\lambda}{\rho} w^*$, it also ends up with a lower interest rate on its savings. Still we cannot conclude on steady-state welfare, since preferences are defined in terms of the parameter s itself.

Secondly, an increase in the capital share at the expense of the wage share makes capital more attractive relatively to children, and induces a higher capital-labor ratio by increasing χ. Wage rate and the interest rate end up higher. Steady-state welfare is thus unequivocally higher the lower the labor share.

Thirdly, an increase in the cost of children ρ makes the desired stock of capital per young increase at steady state[30] due to an increase in the parameter χ, and leaves the relative price of capital unchanged (at q^{-1}). The value of the unique child that the young breeds at steady state also increases. So savings are higher at steady state with a higher ρ parameter, so the wage rate must necessarily be also higher.

Whether w increases more or less than one-for-one with ρ determines the effect on steady-state interest rate on children. In the reasoning above, the value of fertility increases one-for-one while the desired stock of capital increases like ρ^σ in equation (13). In short, when capital and labor are complements, the value of capital held at steady state decreases. Consequently, if $\sigma < 1$ the interest rate decreases at steady state, while it increases if $\sigma > 1$.

Fourthly, the variation of steady-stage wage with λ is unclear but that of *disposable* income $(1-\lambda)w$ is: the more parents can extract from their children' income, the more people will be induced to have children rather than accumulate capital, making further generations poorer – again in terms of disposable income. But disposable income does not tend to 0 as λ tends to one. Otherwise, it would fall behind $\frac{\rho}{(1-\lambda)s}$ which the young needs at steady state to achieve a fertility rate equal to one even if the value of purchased capital falls to zero.

An increase in λ increases desired capital-labor ratio as it increases χ. Steady-state interest rate is increasing with λ when capital and labor are complements. Then, in equation (54), all the effects go in the same direction and an increase in λ increases steady-state interest rate for three reasons: because it increases *ceteris paribus* the interest rate on children (via the term λ), because a higher capital-labor ratio increases steady-state relative capital-labor share, and because a higher λ lowers disposable income of the young $(1-\lambda)w^*$ and further makes old age consumption higher and young age consumption lower at steady state[31]. Consequently, whenever $\sigma < 1$, steady-state interest rate is an increasing function of parameter λ. It is straightforward from equation (54) that it also tends to 0 as λ tends to 0. Steady-state disposable income and interest rate are thus negatively related at steady state and there might exist – in this case – one or several levels for λ that maximize steady-state welfare.

[30]In figure 1, the $\frac{\lambda}{\rho} w$ curve shifts down.

[31]When $\sigma > 1$, first and last effects still work but relative capital share tends to decrease, so that total effect of an increase in λ on equilibrium interest rate is undetermined.

A.2 Proof of lemma (3.1)

Let's denote by f the function:

$$f : x \longmapsto x \left(\frac{\left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho}x\right)^{\zeta-1}}{1 + \left(\frac{s}{1-s}\right)^{\zeta} \left(\frac{\lambda}{\rho}x\right)^{\zeta-1}} \right)^{b} \tag{55}$$

Let $m \equiv \left(\frac{1-s}{s}\right)^{\zeta} \left(\frac{\rho}{\lambda}\right)^{\zeta-1} > 0$. Then we can write function f as:

$$f(x) = x \left(\frac{x^{\zeta-1}}{m + x^{\zeta-1}} \right)^{b}$$

Or again as:

$$f(x) = x^{1-b} \left(\frac{x^{\zeta}}{m + x^{\zeta-1}} \right)^{b} \tag{56}$$

For any value of $\zeta > 0$, both terms are strictly increasing in the right-hand side, and so f is strictly increasing, with $f \xrightarrow{0} 0$ and $f \xrightarrow{\infty} \infty$.

So, for any $(A_t, A_{t+1}, q_{t-1}, q_t, w_t)$ there exists a unique w_{t+1} satisfying equation (42), corresponding to:

$$f(w_{t+1}) = w_t \frac{A_{t+1}}{A_t} \left(\frac{a\chi^{\frac{\sigma-1}{\sigma}} q_t^{\sigma-1} + 1 - a}{a\chi^{\frac{\sigma-1}{\sigma}} q_{t-1}^{\sigma-1} + 1 - a} \right)^{\frac{1-b\sigma}{\sigma-1}} \left(\frac{\rho + \chi q_t^{\sigma-1}}{(1-\lambda)w_t} \right)^{b}$$

A.3 Proof of lemma (3.2)

For fixed values of A and q, call function g:

$$g : x \longmapsto x \left(\frac{\rho + \chi q^{\sigma-1}}{(1-\lambda)x} \right)^{b} = \mu x^{1-b}$$

where $\mu \equiv \left(\frac{\rho+\chi q^{\sigma-1}}{1-\lambda} \right)^{b}$. Function g is clearly strictly increasing and concave.

The equilibrium path is such that $\forall t \geq 0,\ f(w_{t+1}) = g(w_t)$, where function f is defined above in equation (55). It is straightforward from (56) that function $\frac{f}{g}$ can be written:

$$\frac{f}{g}(x) = \frac{1}{\mu} \left(\frac{x^{\zeta}}{m + x^{\zeta-1}} \right)^{b}$$

and so function $\frac{f}{g}$ is strictly increasing, with $\frac{f}{g} \xrightarrow{0} 0$ and $\frac{f}{g} \xrightarrow{\infty} \infty$, for any value of $\zeta > 0$. The generic situation is depicted on figure 9. The two curves are upward-sloping and cross only at $w^*(q)$, so the wage rate monotonically converges to $w^*(q)$ like indicated.

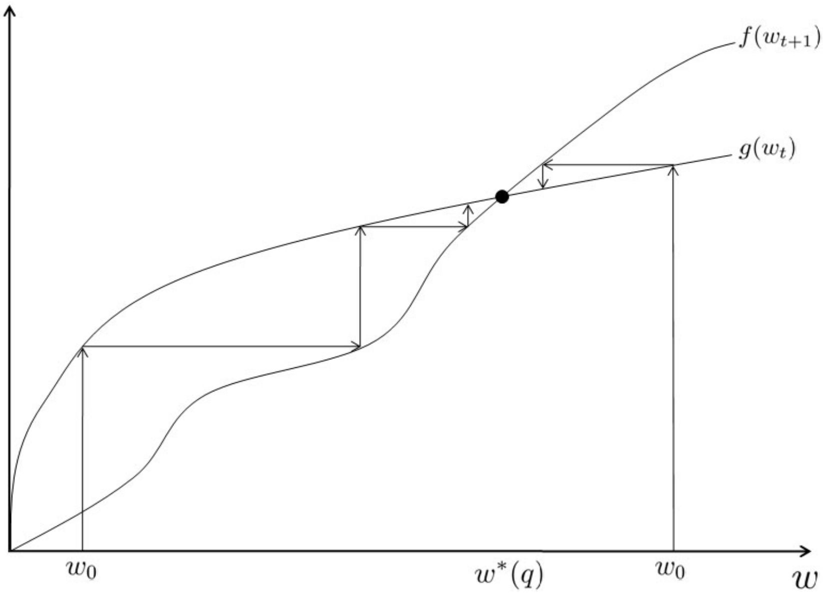

Figure 9: *Convergence to steady-state equilibrium when the utility function is CIES.*

A.4 Proof of proposition (3.1)

With the notations defined in (33) and (35), equation (50) taken to infinity now translates into:

$$1 + z_t \underset{t\infty}{\sim} (1 + z_q)^{1-b\sigma} \left(\frac{\chi q_t^{\sigma-1}}{(1-\lambda) w_t \frac{\psi w_{t+1}^{\zeta-1}}{1+\psi w_{t+1}^{\zeta-1}}} \right)^b \tag{57}$$

which itself, when used at dates t and $t+1$ leads to:

$$1 + z'_t \underset{t\infty}{\sim} \left(\frac{\left(\frac{q_{t+1}}{q_t}\right)^{\sigma-1}}{\frac{w_{t+1}}{w_t}\left(\frac{w_{t+2}}{w_{t+1}}\right)^{\zeta-1}} \frac{1+\psi w_{t+2}^{\zeta-1}}{1+\psi w_{t+1}^{\zeta-1}} \right)^b = \left(\frac{(1+z_q)^{\sigma-1}}{(1+z_t)(1+z_{t+1})^{\zeta-1}} \frac{1+\psi w_{t+2}^{\zeta-1}}{1+\psi w_{t+1}^{\zeta-1}} \right)^b \tag{58}$$

Injecting again the definition of z'_t finally proves that:

$$1 + z'_t \underset{t\infty}{\sim} \left(\frac{(1+z'_t)(1+z_q)^{\sigma-1}}{(1+z_{t+1})^{\zeta}} \frac{1+\psi w_{t+2}^{\zeta-1}}{1+\psi w_{t+1}^{\zeta-1}} \right)^b \tag{59}$$

The case where $\zeta < 1$

When $\zeta < 1$, the term in $\frac{1+\psi \cdot}{1+\psi \cdot}$ is asymptotically equivalent to 1 – shall the wage rate tend to infinity or to a constant[32]. So equation (59) yields:

$$1 + z'_t \underset{t\infty}{\sim} \left(\frac{(1+z_q)^{\sigma-1}}{(1+z_{t+1})^{\zeta}} \right)^{\frac{b}{1-b}} \tag{60}$$

Let's take a final time the same step, and combine (60) at dates t and $t+1$; then:

$$\frac{1+z'_{t+1}}{1+z'_t} \underset{t\infty}{\sim} \frac{1}{(1+z'_{t+1})^{\zeta \frac{b}{1-b}}} \tag{61}$$

so:

$$1 + z'_{t+1} \underset{t\infty}{\sim} (1+z'_t)^{\frac{1}{1+\zeta\frac{b}{1-b}}}$$

Which proves that z'_t tends towards some constant, as the factor $\frac{1}{1+\zeta\frac{b}{1-b}}$ lies in the $(0,1)$ interval. From (61), this constant must be exactly 0. Again, from (60), $1 + z_{t+1}$, the growth rate of the wage rate, tends to $(1+z_q)^{\frac{\sigma-1}{\zeta}}$ as t tends to infinity.

The case where $\zeta > 1$

When $\zeta > 1$, the last term in (59) is now equivalent to $\left(\frac{w_{t+2}}{w_{t+1}}\right)^{\zeta-1} = (1+z_{t+1})^{\zeta-1}$, and so equation (59) at infinitum yields:

$$1 + z'_t \underset{t\infty}{\sim} \left(\frac{(1+z_q)^{\sigma-1}}{1+z_{t+1}} \right)^{\frac{b}{1-b}} \tag{62}$$

The same reasoning as above proves that $1 + z_t$ converges to $(1+z_q)^{\sigma-1}$. ζ then is irrelevant vis-à-vis the asymptotic growth rate of the wage rate.

References

[1] R. C. Allen (2001) "The great divergence: Wages and prices from the Middle Ages to the First World War," *Explorations in Economic History,* Vol. 38 (Oct.), pp. 411–447.

[32] I do not investigate here the possibility of permanent fluctuations in the wage rate.

[2] R. C. Allen (2009) "Engels' pause: Technical change, capital accumulation, and inequality in the british industrial revolution," *Explorations in Economic History,* Vol. 46, pp. 418–435.

[3] L. Angeles (2008) "GDP per capita or real wages? Making sense of conflicting views on pre-industrial Europe," *Explorations in Economic History,* Vol. 45, pp. 147–163.

[4] K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow (1961) "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics,* Vol. 43, No. 3 (Aug.), pp. 225–250.

[5] Q. Ashraf and O. Galor (2011) "Dynamics and Stagnation in the Malthusian Epoch," *American Economic Review,* No. 239.

[6] G. S. Becker, K. M. Murphy and R. Tamura (1990) "Human Capital, Fertility, and Economic Growth," *Journal of Political Economy,* Vol. 98, No. 5 (Oct.), part 2, pp. S12–S37.

[7] G. S. Becker, E. L. Glaeser and K. M. Murphy (1999) "Population and Economic Growth," *American Economic Review Papers and Proceedings,* Vol. 89, No. 2 (May), pp. 145–149.

[8] G. Clark (2005) "The Condition of the Working Class in England: 1209–2004," *Journal of Political Economy,* Vol. 113, No. 6 (Dec.), pp. 1307–1340.

[9] G. Clark (2007) *A Farewell to Alms: A Brief Economic History of the World,* Princeton University Press, Princeton.

[10] G. Clark (2009) "The Macroeconomic Aggregates for England, 1209-2008," *UC Davis Working Paper 09-19.*

[11] W. J. Collins and J. G. Williamson (2001) "Capital-Goods Prices and Investment, 1870–1950," *Journal of Economic History,* Vol. 61, No. 1 (Mar.), pp. 59–94.

[12] I. G. Conde-Ruiz, E. L. Giménez and M. Pérez-Nievas (2010) "Millian Efficiency with Endogenous Fertility," *Review of Economic Studies,* 77, pp. 154–187.

[13] N. F. R. Crafts (1985) *British Economic Growth during the Industrial Revolution,* Clarendon Press, Oxford.

[14] O. de La Grandville (2009) *Economic Growth – A Unified Approach,* Cambridge University Press, Cambridge.

[15] O. Galor (2005) "From Stagnation to Growth: Unified Growth Theory," in *Handbook of Economic Growth, vol. 1A* (P. Aghion and S. N. Durlauf, eds.), Elsevier, Amsterdam: chapter 4, pp. 171–293.

[16] O. Galor and O. Moav (2004) "From Physical to Human Capital Accumulation: Inequality and the Process of Development," *Review of Economic Studies,* Vol. 71, No. 4 (Oct.), pp. 1001–1026.

[17] O. Galor and O. Moav (2006) "Das Human-Kapital: A Theory of the Demise of the Class Structure," *Review of Economic Studies,* Vol. 73, No. 1 (Jan.), pp. 85–117.

[18] O. Galor and D. N. Weil (1999) "From Malthusian Stagnation to Modern Growth," *American Economic Review Papers and Proceedings,* Vol. 89, No. 2 (May), pp. 150–154.

[19] O. Galor and D. N. Weil (2000) "Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond," *American Economic Review,* Vol. 90, No. 4 (Sep.), pp. 806–828.

[20] G. D. Hansen and E. C. Prescott (2002) "Malthus to Solow," *American Economic Review,* Vol. 92, No. 4, pp. 1205–1217.

[21] C. I. Jones (1994) "Economic Growth and the Relative Price of Capital," *Journal of Monetary Economics,* Vol. 34, pp. 359–382.

[22] C. I. Jones (1999) "Was an Industrial Revolution Inevitable? Economic Growth Over the Very Long Run," *NBER Working Paper No. 7375.*

[23] M. Kremer (1993) "Population growth and technological change: One million B.C. to 1990," *Quarterly Journal of Economics*, Vol. 108, No. 3 (Aug.), pp. 681–716.

[24] N. Lagerlof (2006) "The Galor-Weil model revisited: A quantitative exploration," *Review of Economic Dynamics*, 9.

[25] R. E. Lucas (2002) *The Industrial Revolution: Past and Future,* Harvard University Press, Cambridge, MA.

[26] A. Maddison (2001) *The World Economy: A Millennia Perspective,* OECD, Paris.

[27] K. H. O'Rourke, A. S. Rahman and A. M. Taylor (2008) "Luddites and the Demographic Transition," *NBER Working Paper No. 14484.*

[28] K. Pomeranz (2000) *The Great Divergence – China, Europe, and the Making of the Modern World Economy,* Princeton University Press, Princeton.

[29] A. S. Rahman (2011) "A Simple Theory on the Effects of Industrialization," *mimeo.*

[30] L. K. Raut and T. N. Srinivasan (1994) "Dynamics of Endogenous Growth," *Economic Theory,* Vol. 4, No. 5 (Aug.), pp. 777–790.

[31] H.-J. Voth (2003) "Living standards during the Industrial Revolution: An economist's guide," *American Economic Review,* Vol. 93, No. 2 (May), pp. 221–226.

[32] J. L. Weisdorf (2004) "From stagnation to growth: Revisiting three historical regimes," *Journal of Population Economics,* Vol. 17, pp. 455–472.

[33] J. L. Weisdorf (2008) "Malthus revisited: Fertility decision making based on quasi-linear preferences," *Economics Letters,* Vol. 99, pp. 127–130.