# A Model of Endogenous growth - The Case of an Innovative Economy.

Constantin Chilarescu*,1 and Ioana Viasu 2

[1] *Laboratoire CLERSE Université de Lille* 1*, France*
*E-mail: Constantin.Chilarescu@univ-lille1.fr*
[2] *Faculty of Economics and Business Administration, West University of Timisoara, Romania*

**Abstract**

The main purpose of our paper is to give some clarifications to the endogenous growth model with physical capital, human capital and $R\&D$, developed by Funke and Strulik, having as starting point the basic model proposed by Grossman and Helpman.

Keywords: Endogenous growth model; innovation; education.
JEL Classifications: $C61$, $C62$, $O41$.

# 1 Introduction

A large number of papers has been published in the last years on this subject, following the line developed by Grossman and Helpman [7, 1991]. Among them, to our knowledge, the first one is the paper of Eriksson [4, 1996] where he work out a model that is only a slight modification of those developed by Grossman and Helpman.

Few years later, the balanced growth path of the endogenous growth model with physical capital, human capital and $R\&D$ has been explored by Funke and Strulik [5, 2000] (henceforth FS), and then by Arnold [1, 2000]. $FS$ suggest that the typical advanced economy follows three development

*Corresponding author

phases, characterized in a temporal order by: physical capital accumulation, human capital formation and innovation.

Gomez [6, 2005] analyzes the equilibrium dynamics of this model and correct the analysis of $FS$ and Arnold. Sequeira [11, 2008] incorporates an erosion effect into this model and claims that this effect significantly improves the fit between the model and the data. Iacopetta [8, 2010], [9, 2011] extends the earlier analysis of $FS$ and argues that other sequences of the phases of development are possible and shows that this model can generate trajectories in which innovation precedes human capital formation. This trajectory accords with the observation that the rise in formal education followed with a considerable lag the process of industrialization.

In their paper, $FS$ assume that consumption goods, investment goods and intermediate goods are all produced with the same technology and can be transformed one to one without further costs from output of the industrial sector, which is produced under a Cobb-Douglas technology. In fact, only invention of new intermediates is determined solely by the aggregate knowledge devoted to $R\&D$. Intermediate goods are produced by labor alone, under monopolistic competition and therefore, in order to fulfill the conditions for market clearing, we need to consider this kind of labor in the model.

In order to obtain our results, we follow the line developed by Grossman and Helpman [7, 1991], more exactly, the model proposed in Chapter 5. They consider the case of an economy where capital goods and intermediate inputs each have only a single use and the final output is either consumed or invested. In their model, labor is used in $R\&D$, in the production of intermediate and final goods.

We concentrate our analysis only on the case of an innovative economy and the paper is organized as follows. In the next section we argue and develop the differential equations that describe the dynamics of all variables of the model. The third section gives a complete characterization of the balanced growth path. Section four studies the stability conditions and the final section presents some conclusions and numerical simulations.

# 2 A model of growth with innovation

In this section we develop a model of endogenous growth with innovation and derive the differential equations that describe the dynamics of the economy. Without loss of generality, we suppose that the economy is populated by a large and constant number of identical agents, normalized to one, so that all the variables can be interpreted as per capita quantities. Each individual is endowed with one unit of time. We assume that innovative products are intermediate inputs into the production of a single, final good. The final good $y$ can be consumed by the households or purchased by firms as physical capital. The technology for producing final output requires, besides intermediates, as inputs, labor and physical capital. Intermediates are produced by labor alone, and labor is the sole input into $R\&D$. Final output $y$ is produced according to a Cobb-Douglas production function

$$y = \gamma k^{\beta} d^{\eta} h_k^{1-\beta-\eta}, \tag{1}$$

where $\gamma$ is a positive constant, $\beta$ and $\eta$ are positive elasticity parameters with $\beta + \eta \leq 1$, $k$ is physical capital, $h_k$ denotes the share of human capital employed to produce the final good and $d$ represents an aggregate index of intermediate goods. Grossman and Helpman have adopted an integral alternative of the aggregate index of intermediate goods, but this index can create some difficulties when we determine the derivative of it with respect to a variable $x(i)$, and decide to replace it. More precisely, we prefer to use the original specification of Dixit and Stiglitz [3, 1977].

$$d\left(x_1, x_2, \cdots, x_n\right) = \left[\sum_{i=1}^{n} x_i^{\alpha}\right]^{\frac{1}{\alpha}}, \tag{2}$$

where $n = n(t)$ is a measure of products invented before time $t$ and can be considered as the number of available varieties. $x_i$ represents the input of component $i$ in the production of final good, and $0 < \alpha < 1$ controls the elasticity of substitution between intermediates, $\varepsilon = \frac{1}{1-\alpha}$. We also assume that the number of products invented is solely determined by the share of human capital devoted to the $R\&D$ activity, denoted by $h_d$:

$$\dot{n} = \delta h_d, \tag{3}$$

where $\delta > 0$ is an efficiency parameter. Furthermore, each individual can spend a part of his time in the education sector, to rise the human capital level and we suppose that this process is proportional to the share of human capital devoted to education, denoted here by $h_e$

$$\dot{h} = \xi h_e, \tag{4}$$

where $\xi > 0$ is an efficiency parameter.
The market for the final good $y$ is assumed to be perfectly competitive and therefore the price of this good $p_y$ equals its marginal production cost. Neglecting depreciation, the economy's resource constraint will be given by

$$\dot{k} = y - c - \sum_{i=1}^{n} x_i p_i, \tag{5}$$

where $p_i$ is the price of an intermediate expressed in terms of $y$ (see Barro and Sala-i-Martin [2, 2004], pag. 288). This price will be determined later in the paper. The derivative of function $d$, defined by $Eq.$ (2), with respect to $x_j$ is given by

$$\frac{d}{dx_j}[d] = d^{1-\alpha} x_j^{\alpha-1}, j = 1, 2, \cdots, n, \tag{6}$$

and therefore

$$p_d^j = d^{1-\alpha} x_j^{\alpha-1} p_d, j = 1, 2, \cdots, n, \tag{7}$$

is the price of intermediate $x_j$ in terms of $d$, where $p_d$ is the unit price of $d$. By assuming in a formal way that the marginal and average cost of production is constant, normalized to one, the lemma of Shephard provides the following prices, in terms of $y$

$$p_d = \eta \frac{y}{d},\ r = \beta \frac{y}{k},\ w_k = (1 - \beta - \eta) \frac{y}{h_k}, \tag{8}$$

where $w_k$ is the price of labor employed for the production of capital good. In the same terms, the price of any intermediate $x_j$, denoted here by $p_j$, will then be given by

$$p_j = \frac{\eta y}{d^\alpha} x_j^{\alpha-1}. \tag{9}$$

This price is paid by the final output sector, for a unit of $x_j$, to monopolist producer. As we can observe, this price is dependent on $x_j$. Therefore, the demand function for an intermediate $x_j$ at price $p_j$ will be

$$x_j = \left[\frac{d^\alpha}{\eta y}\right]^{\frac{1}{\alpha-1}} p_j^{\frac{1}{\alpha-1}}. \tag{10}$$

According to the technological hypothesis, production of the final output requires intermediates and assuming that intermediates are produced by labor alone, and labor is the sole input into $R\&D$, facing the demand function (10), the monopolist supplier of variety $x_j$ maximizes operating profits

$$\pi_j = (p_j - w_j)\, x_j, \tag{11}$$

where $w_j$ is the price of labor used to produce one unit of intermediate. The equilibrium on the labor market requires that $w_j = w_k$ and we denote by $w$ their common value. Substitution of $x_j$ from $Eq$. (10) in to $Eq$. (11), provides this optimal price

$$p_j = \frac{w}{\alpha}. \tag{12}$$

To simplify, we denote by $p_x$ this unique price. In a symmetric equilibrium, the quantity supplied for all intermediates is the same, that is $x_j = x$ and therefore

$$d = A_d nx = n^{\frac{1-\alpha}{\alpha}} nx. \tag{13}$$

where $A_d$ represents the factor productivity of intermediates. If $X$ denotes the aggregate volume of intermediate output, then $dp_d = Xp_x$ implies

$$p_d = \frac{p_x}{A_d}. \tag{14}$$

This pricing strategy yields the total quantity of intermediates required by the sector of production

$$nx = \frac{\alpha\eta y}{w}. \tag{15}$$

Any intermediate good can be produced using one unit of labor. Then, regardless of its composition, $X = nx$ measures the resources embodied in final goods, that is the quantity of labor incorporated

$$nx = \frac{\alpha\eta}{1-\beta-\eta} h_k. \tag{16}$$

Therefore, the total quantity of labor employed to produce the final good, including interediates is then given by (see Grossman and Helpman cited paper, pag. 119):

$$h_y = \frac{\alpha\eta}{1-\beta-\eta} h_k + h_k = \frac{1-\beta-\eta(1-\alpha)}{1-\beta-\eta} h_k, \tag{17}$$

from where we get

$$h_k = \frac{1-\beta-\eta}{1-\beta-\eta(1-\alpha)} h_y, \tag{18}$$

and therefore, $Eq.$ (16) becomes

$$nx = \frac{\alpha\eta}{1-\beta-\eta(1-\alpha)} h_y. \tag{19}$$

Successively substitution of $Eqs$ (10), (12) and (13) into $Eq.$ (11), provides a profit equal to

$$\pi = \frac{\eta(1-\alpha)}{n} y. \tag{20}$$

If $\nu$ is the market price of an intermediate, then, in a general equilibrium, free entry into $R\&D$ requires

$$w = \delta\nu \quad \Leftrightarrow \quad g_\nu = \frac{\dot{\nu}}{\nu} = \frac{\dot{w}}{w} = g_w. \tag{21}$$

The equilibrium in the capital market requires that the interest rate equals the dividend rate $\frac{\pi}{\nu}$ plus the rate of capital gain $\frac{\dot{\nu}}{\nu}$ and thus we can write

$$g_\nu = \frac{\dot{\nu}}{\nu} = r - \frac{\pi}{\nu} = g_w.$$

Substituting $\pi$ from $Eq.$ (20), $\nu$ from $Eq.$ (21), $w$ from $Eq.$ (8) and $h_k$ from $Eq.$ (18), we obtain

$$g_w = r - \frac{\delta\eta(1-\alpha)}{1-\beta-\eta(1-\alpha)} \frac{h_y}{n}. \tag{22}$$

Substituting $p_x$ from $Eq.$ (12) and $nx$ from $Eq.$ (15), we can rewrite the economy's resource constraint (5) as

$$\dot{k} = (1-\eta)y - c. \tag{23}$$

Substitution of $d$ from $Eq.$ (13), $nx$ from $Eq.$ (19) and $h_k$ from $Eq.$ (18) into $Eq.$ (1) provides the final form of production function

$$y = Ak^\beta h_y^{1-\beta} n^{\frac{\eta(1-\alpha)}{\alpha}}, \; A = \frac{\gamma\,(\alpha\eta)^\eta\,(1-\beta-\eta)^{1-\beta-\eta}}{[1-\beta-\eta(1-\alpha)]^{1-\beta}}. \tag{24}$$

and the two prices, of the total labor employed to produce the final good and of physical capital

$$w = (1-\beta)\frac{y}{h_y} \quad r = \beta\frac{y}{k}. \tag{25}$$

Let $(u_y, u_d, u_e)$ be the amount of time allocated to produce the final good, to innovation and respectively to education. Then, we have: $h_y = hu_y$, $h_d = hu_d$ and $h_e = hu_e$ and the full employment requires:

$$u_y + u_d + u_e = 1. \tag{26}$$

The next equation describes the dynamics of the budget constraint

$$\dot{a} = ra + w\left(1 - u_e\right)h - c. \tag{27}$$

Subject to the budget constraint *Eq.* (27), to the development of skill *Eq.* (4) and using as state variables $(h, a)$ and as control variables $(c, u_e)$, we can write the following optimization problem

$$\max_{\{c,u_e\}} \int_0^\infty \frac{c^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt, \tag{28}$$

where $\rho > 0$ denotes the time preference rate and $\theta > 0$ denotes the intertemporal elasticity of substitution. The Hamiltonian is given by

$$J = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda\left[ra + w\left(1 - u_e\right)h - c\right] + \mu\xi h u_e. \tag{29}$$

The first derivatives of Hamiltonian with respect to $c$, $u_e$, $h$ and $a$ provide the following system of differential equations

$$\begin{cases} g_c = \frac{\dot{c}}{c} = \frac{r-\rho}{\theta}, \\ g_w = \frac{\dot{w}}{w} = r - \xi, \\ g_\mu = \frac{\dot{\mu}}{\mu} = \rho - \xi, \\ g_\lambda = \frac{\dot{\lambda}}{\lambda} = \rho - r. \end{cases} \tag{30}$$

where $g_z$ denotes the growth rate of variable $z$. Combining the second equation of the above system, with *Eq.* (22), we can determine the following relation for $u_y$

$$u_y = \frac{\xi\left[1 - \beta - \eta(1-\alpha)\right]}{\delta\eta(1-\alpha)} \frac{n}{h}, \tag{31}$$

from where it immediately follows

$$g_{u_y} = g_n - g_h. \tag{32}$$

After log-differentiating *Eq.* (24) with respect to time, the output growth rate is then determined by

$$g_y = \beta g_k + (1-\beta)\left(g_h + g_{u_y}\right) + \frac{\eta(1-\alpha)}{\alpha} g_n, \tag{33}$$

and substitution of the above result provides

$$g_y = \beta g_k + \frac{\alpha(1-\beta) + \eta(1-\alpha)}{\alpha} g_n, \tag{34}$$

The *Eqs.* (3) and (23) can also be written

$$g_n = \delta \frac{h}{n} u_d, \tag{35}$$

and

$$g_k = \frac{1-\eta}{\beta} r - \frac{c}{k}. \tag{36}$$

Log-differentiation of $r$ and $w$ from *Eq.* (25) provides

$$g_r = -\frac{(1-\beta)(1-\eta)}{\beta} r + (1-\beta)\frac{c}{k} + \frac{\alpha(1-\beta) + \eta(1-\alpha)}{\alpha} g_n \tag{37}$$

and

$$g_w = g_y - \left(g_h + g_{u_y}\right) = g_y - g_n \ \Rightarrow g_n = g_y - g_w, \tag{38}$$

and after some algebraic manipulations yields

$$g_n = \frac{\alpha}{\eta(1-\alpha) - \alpha\beta}\left[\eta r + \beta \frac{c}{k} - \xi\right]. \tag{39}$$

Substituting *Eq.* (39) into the *Eq.* (37) and denoting by $\chi = \frac{c}{k}$ and $\psi = \frac{n}{h}$, enable us to write down the following system of differential equations in terms of $(\chi, r, \psi)$:

$$\begin{cases} g_\chi = \frac{\beta - \theta(1-\eta)}{\beta\theta} r + \chi - \frac{\rho}{\theta}, \\ g_r = A_1 r + A_2 \chi - A_3, \\ g_\psi = g_n + \frac{\xi}{\delta}\left(B_2 + g_n\right)\psi - \xi, \\ g_n = B_1\left[\eta r + \beta\chi - \xi\right], \end{cases} \tag{40}$$

where

$$A_1 = \frac{\alpha\beta(1-\beta) - \eta(1-\alpha)(1-\beta-\eta)}{\beta\left[\eta(1-\alpha) - \alpha\beta\right]},$$

$$A_2 = \frac{\eta(1-\alpha)}{\eta(1-\alpha) - \alpha\beta},\ A_3 = \frac{\xi\left[\alpha(1-\beta) + \eta(1-\alpha)\right]}{\eta(1-\alpha) - \alpha\beta},$$

$$B_1 = \frac{\alpha}{\eta(1-\alpha) - \alpha\beta},\ B_2 = \frac{\xi\left[1-\beta-\eta(1-\alpha)\right]}{\eta(1-\alpha)}.$$

To understand the dynamics of our system, a supplementary differential equation will be necessary, that is the first equation of system (30).

A remark is absolutely necessary here.

**Remark 1** *The functions $r$ and $w$ are independent of variables $h$ and $u_y$.*

**Proof of Remark** 1. Substituting *Eq.* (31) into *Eqs.* (25) yield

$$r = A_r k^{\beta-1} n^{\frac{\alpha(1-\beta)+\eta(1-\alpha)}{\alpha}} \quad w = A_w k^{\beta} n^{\frac{\eta(1-\alpha)-\alpha\beta}{\alpha}},$$

where

$$A_r = \beta A \left\{\frac{\xi\left[1-\beta-\eta(1-\alpha)\right]}{\delta\eta(1-\alpha)}\right\}^{1-\beta},$$

and

$$A_w = (1-\beta) A \left\{\frac{\xi\left[1-\beta-\eta(1-\alpha)\right]}{\delta\eta(1-\alpha)}\right\}^{-\beta}.$$

Now it is clear why in the above optimization problem the first derivatives of Hamiltonian with respect to variables $h$ and $u_e$ do not act on the two functions $w$ and $r$.

# 3 Balanced growth equilibrium

In this section, we focus our analysis on the balanced growth equilibrium, characterized by the fact that all variables grow at constant, but possible different rates, and the shares of human capital in its different uses are constant. The system described above reaches the balanced growth path ($BGP$) if there exists $t_*$ (possibly infinite), such that for all $t \geq t_*$, $g_{u_y} = g_{u_d} = g_{u_e} = 0$ and $g_k = g_c = g_y = g \neq g_h = g_n$, where $g_z$ denotes the growth rate of variable $z$. The following proposition gives our first main result that characterize the balanced growth path.

**Proposition 1** *Let* $\xi > \rho$, $\alpha > \alpha_1$ *and* $\theta > \theta_m$. *Then the above system reaches the BGP and the following statements are valid*

$$r_* = \frac{\theta\xi\left[\alpha(1-\beta)+\eta(1-\alpha)\right]-\eta\rho(1-\alpha)}{\theta\left[\alpha(1-\beta)+\eta(1-\alpha)\right]-\eta(1-\alpha)} > \xi, \tag{41}$$

$$\chi_* = \frac{\left[\theta(1-\eta)-\beta\right]r_*+\beta\rho}{\beta\theta} > 0, \tag{42}$$

$$g_{n_*} = \frac{\alpha(1-\beta)(\xi-\rho)}{\alpha\theta(1-\beta)+\eta(1-\alpha)(\theta-1)} > 0, \tag{43}$$

$$\psi_* = \frac{\delta\left(\xi-g_{n_*}\right)}{\xi\left(B2+g_{n_*}\right)} > 0, \tag{44}$$

$$u_{y_*} = \frac{1-\beta-\eta(1-\alpha)}{\eta(1-\alpha)}\frac{\xi-g_{n_*}}{B2+g_{n_*}} > 0, \tag{45}$$

$$u_{d_*} = \frac{\psi_* g_{n_*}}{\delta} > 0, \tag{46}$$

$$u_{e_*} = \frac{g_{n_*}}{\xi} > 0, \tag{47}$$

$$g_* = \frac{r_*-\rho}{\theta} = \frac{\left[\alpha(1-\beta)+\eta(1-\alpha)\right](\xi-\rho)}{\alpha\theta(1-\beta)+\eta(1-\alpha)(\theta-1)} > 0, \tag{48}$$

*where*

$$\theta_m = \frac{1}{\xi}\left[\frac{\eta\rho(1-\alpha)}{\alpha(1-\beta)+\eta(1-\alpha)}+\frac{\beta(\xi-\rho)}{1-\eta}\right]$$

*and*

$$\alpha_1 = \frac{\eta(1-\beta-\eta)}{\eta(1-\beta-\eta)+\beta(1-\beta)}.$$

**Proof of Proposition** 1. According to *Eq.* (32), constancy of $u_y$ implies that, at equilibrium, the growth rate of $n$ equals the growth rate of $h$, that is, $g_{n*} = g_{h*}$ and therefore $g_{\psi*} = 0$. Constancy of $g_{c_*}$ implies the constancy of $r_*$, i.e., $g_{r_*} = 0$. Therefore $g_{y_*} = g_{k_*}$, and thus $\chi$ is also constant in the steady state, i.e., $g_{\chi_*} = 0$ and therefore we have $g_* = g_{y_*} = g_{k_*}$, where we denote by $g_{z*}$ the equilibrium's growth rate of variable $z$. Positivity of $r_*$ and $r_* > \xi$ is ensured for any

$$\theta > \frac{\eta(1-\alpha)}{\alpha(1-\beta)+\eta(1-\alpha)} = \theta_1 < \theta_m$$

for all $\alpha > \alpha_1$. A sufficient condition that ensure the positivity of $\chi_*$ is $\theta(1-\eta)-\beta > 0$, but this condition is too restrictive. By direct computation we obtain that $\chi_* > 0$ for all $\theta > \theta_m$. Because $u_{y_*} > 0$, $u_{d_*} > 0$, $u_{e_*} > 0$ and $u_{y_*} + u_{d_*} + u_{e_*} = 1$ we obviously have $\{u_{y_*}, u_{d_*}, u_{e_*},\} \in (0,1)$. The other relations follow immediately by direct computation and thus the proof is completed.

# 4 Stability property of the $BGP$

In this section we investigate the stability properties of the $BGP$ found in the previous section. For our analysis we need to consider only the first three equations of (40):

$$\begin{cases} \dot{r} = \left[A_1 r + A_2\chi - A_3\right] r = f_1, \\ \\ \dot{\chi} = \left[\frac{\beta-\theta(1-\eta)}{\beta\theta} r + \chi - \frac{\rho}{\theta}\right] \chi = f_2, \\ \\ \dot{\psi} = \left[g_n + \frac{\xi}{\delta}\left(B_2 + g_n\right)\psi - \xi\right]\psi = f_3. \end{cases} \tag{49}$$

We will prove that the competitive equilibrium solution is locally unique, i.e., the $BGP$ is determinate, if the Jacobian of the reduced system has at least one eigenvalue with negative real part. The study of stability equilibrium, *i.e.* around the $BGP$, has nothing to do with the fact that we know or we do not know the starting values of some variables. All what we need to know are their values at $BGP$ and the eigenvalues of Jacobian. This assertion is true because the equilibrium point is unique and known, and this claim was proved in the previous section.

There are two elements that are irrelevant for our analysis. We denote these two elements by $a$ and respectively by $b$. The Jacobian evaluated at $BGP$ is given by:

$$J_* = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & 0 \\ a & b & J_{33} \end{pmatrix} \tag{50}$$

where:

$$J_{11} = A_1 r_*,\ J_{12} = A_2 r_*,\ J_{21} = \frac{\beta-\theta(1-\eta)}{\beta\theta}\chi_*,$$
$$J_{22} = \chi_*,\ J_{33} = \xi - g_{n_*} > 0.$$

The following proposition gives the necessary and sufficient conditions which ensure the local saddle-point stability of the $BGP$.

**Proposition 2** *Let $\xi > \rho$, $\alpha > \alpha_1$ and $\theta > \theta_m$. Then the following statements are valid*

*i. If $\alpha > \alpha_m$, then the $BGP$ is a saddle-point equilibrium, where*

$$\alpha_m = \frac{\eta}{\eta + \beta} > \alpha_1.$$

*ii. If $\alpha_1 < \alpha < \alpha_m$, then the $BGP$ is an unstable equilibrium.*

**Proof of Proposition** 2. The matrix $J_*$ has as an immediate property, the fact that two of its eigenvalues, are the eigenvalues of the matrix $J1_*$.

$$J1_* = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

and the third eigenvalue of $J_*$ is equal to $\xi - g_{n_*} > 0$. By direct computation we obtain

$$Det(J1_*) = \frac{\alpha\theta(1-\beta) + \eta(1-\alpha)(\theta-1)}{\theta\left[\eta(1-\alpha) - \alpha\beta\right]} r_* \chi_* \text{ and } Tr(J1_*) = A_1 r_* + \chi_*,$$

where $Det(J1_*)$ is the determinant of $J1_*$ and $Tr(J1_*)$ is the trace of $J1_*$. In order to understand the trajectories of variables in the neighborhood of $BGP$ we introduce the following notations: $z_1 = r - r_*$, $z_2 = \chi - \chi_*$ and $z_3 = \psi - \psi_*$. Therefore we can write:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & 0 \\ a & b & J_{33} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}. \tag{51}$$

Let $\omega_1$, $\omega_2$, $\omega_3$ be the three eigenvalues of Jacobian $J$. The solutions of the above system can therefore be written

$$\begin{cases} r(t) = r_* + \varphi_1\pi_{11}e^{\omega_1(t-t_*)} + \varphi_2\pi_{12}e^{\omega_2(t-t_*)} + \varphi_3\pi_{13}e^{\omega_3(t-t_*)}, \\ \chi(t) = \chi_* + \varphi_1\pi_{21}e^{\omega_1(t-t_*)} + \varphi_2\pi_{22}e^{\omega_2(t-t_*)} + \varphi_3\pi_{23}e^{\omega_3(t-t_*)}, \\ \psi(t) = \psi_* + \varphi_1\pi_{31}e^{\omega_1(t-t_*)} + \varphi_2\pi_{32}e^{\omega_2(t-t_*)} + \varphi_3\pi_{33}e^{\omega_3(t-t_*)}, \end{cases} \tag{52}$$

where $\pi_{ij}, i, j = 1, 2, 3$ are constant elements of the eigenvectors and $\varphi_1$, $\varphi_2$, $\varphi_3$ are constants to be determined. First observe that $\omega_3 > 0$, and therefore this equilibrium cannot be asymptotically stable. What we need now is to take if $\varphi_3 = 0$ to ensure the existence of a saddle-path stability. Under this hypothesis, the above system becomes

$$\begin{cases} r(t) = r_* + \varphi_1 \pi_{11} e^{\omega_1 (t-t_*)} + \varphi_2 \pi_{12} e^{\omega_2 (t-t_*)}, \\ \chi(t) = \chi_* + \varphi_1 \pi_{21} e^{\omega_1 (t-t_*)} + \varphi_2 \pi_{22} e^{\omega_2 (t-t_*)}, \\ \psi(t) = \psi_* + \varphi_1 \pi_{31} e^{\omega_1 (t-t_*)} + \varphi_2 \pi_{32} e^{\omega_2 (t-t_*)}. \end{cases} \tag{53}$$

The necessary and sufficient conditions which ensure this kind of stability are achieved if at least one of the remaining two eigenvalues, has a negative real part, or equivalently, $Det(J1_*) < 0$. The numerator of $Det(J1_*)$ is positive and therefore, what we need is $\eta(1-\alpha) - \alpha\beta < 0$, that is $\alpha > \alpha_m$.

If $Det(J1_*) > 0$ it immediately follows that the real parts of the two eigenvalues have the same sign. This is true for any value $\alpha_1 < \alpha < \alpha_m$. Analyzing the trace of Jacobian we obtain $Tr(J1_*) > 0$ and consequently the two eigenvalues can have only positive real parts, and thus the proof is completed.

We conclude this section by noting that:

- Some of our results differ from those of the above cited authors.
- Our model clarifies some questionable aspects of the model developed by $FS$, Arnold and later analyzed by Manuel Gomez, Iacopetta and Tiago Sequeira.

# 5 Conclusions

In this paper we explored the equilibrium dynamics of an innovative economy, via an endogenous growth model with physical capital, human capital and $R\&D$. In the second section we developed a model of endogenous growth with innovation and derive the differential equations that describe the dynamics of the economy. We proved in the third section that, under general fairly conditions, the model reaches the balanced growth path and determined the values of all variables at $BGP$. Finally, the previous section was dedicated to investigate the stability properties of the $BGP$.

The main results proposed by our paper are given in Propositions 1 and 2. As we can observe, these results make some light on the general properties of the model. First observe that, in our model, $\theta$ does not have un upper bound as it is the case of some cited papers.

We close this final section with some numerical simulations in order to confirm the theoretical aspects presented in our paper. $FS$ have used in their numerical simulation section, the data available in the case of the US - economy, from the dataset compiled by Jorgenson and Fraumeni [10, 1993]. Another remark is necessary here. As it was observed by Jorgenson and Fraumeni (see cited paper, page 17), analysing the $US-$economy for the period $1973-1986$, physical capital input is the most important source of growth and therefore $\eta<\beta$ seems to be evident. For our simulation procedure we consider the following benchmark values:

a. $\beta=0.25$, $\eta=0.20$, $\xi=0.05$, $\delta=0.1$, $\rho=0.03$, $A=1$, $\theta=2$, $\alpha=0.5$. This parametrization yields the following equilibrium:

   $r_*=0.0523$, $\chi_*=0.1564$, $\psi_*=0.2467$, $g_*=0.0111$, $g_{n_*}=0.0088$, $u_{y_*}=0.8018$, $u_{d_*}=0.0218$, $u_{e_*}=0.1764$ and the eigenvalues are: $\omega_1=0.2983$, $\omega_2=-0.4665$ and $\omega_3=0.0412$. As we can observe, in this case, the equilibrium is saddle-path stable.

b. $\beta=0.25$, $\eta=0.20$, $\xi=0.05$, $\delta=0.1$, $\rho=0.03$, $A=1$, $\theta=2$, $\alpha=0.4$. This parametrization yields the following equilibrium:

   $r_*=0.0533$, $\chi_*=0.1590$, $\psi_*=0.3077$, $g_*=0.0117$, $g_{n_*}=0.0083$, $u_{y_*}=0.8077$, $u_{d_*}=0.0256$, $u_{e_*}=0.1667$ and the eigenvalues are: $\omega_1=0.1275+0.3693i$, $\omega_2=0.1275-0.3693i$ and $\omega_3=0.0417$. As we can observe, in this case, the equilibrium is unstable.

# References

[1] Arnold, L., 2000. Endogenous growth with physical capital, human capital and product variety: A comment, European Economic Review, 44, $1599-1605$.

[2] Barro, R.J. and Sala-i-Martin, X., 2004. Economic Growth, The MIT Press, 2nd ed.

[3] Dixit, A. and Stiglitz, J., 1977. Monopolistic competition and optimum product diversity, American Economic Review, 67, $297 - 308$.

[4] Eriksson, C., 1996. Market failures in the $R\&D$ growth model withl endogenous labor supply, Journal of Public Economics, 61, $445 - 454$.

[5] Funke, M. and Strulik, H., 2000. On endogenous growth with physical capital, human capital and product variety, European Economic Review, 44, $491 - 515$.

[6] Gomez, M.A., 2005. Transitional dynamics in an endogenous growth model with physical capital, human capital and $R\&D$, Studies in Nonlinear Dynamics & Econometrics, 9 (1), Article 5.

[7] Grossman, G.M. and Helpman, E., 1991. Innovation and Growth in the Global Economy, Cambridge, MA: MIT Press.

[8] Iacopetta, M., 2010. Phases of economic development and the transitional dynamics of an innovation - education growth model, European Economic Review, 54, $317 - 330$.

[9] Iacopetta, M., 2011. Formal education and public knowledge, Journal of Economic Dynamics and Control, 35, $676 - 693$.

[10] Jorgenson, D.W. and Fraumeni, B.M., 1993. Education and productivity growth in a market economy, Atlantic Economic Journal, 21, (2), $1 - 25$.

[11] Sequeira, T.N., 2008. Transitional dynamics of an endogenous growth model with an erosion effect, Manchester School, 76, (4), $436 - 452$.