Integrated Modified OLS estimation of cointegrating regressions with trending regressors and residual-based tests for the null of cointegration

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Abstract:
In this paper we discuss the asymptotically efficient estimation of a univariate static cointegrating regression relationship when we take into account the deterministic structure of their stochastically integrated components, in a slightly more general framework that considered by Hansen (1992). After reviewing the properties of OLS and Fully Modified OLS (FM-OLS) estimation in this framework, we consider the recently proposed Integrated Modified OLS (IM-OLS) estimator by Vogelsang and Wagner (2011) of the cointegrating vector and propose a new proper specification of the integrated modified cointegrating regression equation. This alternative method of bias removal has the advantage over the existing methods that does not require any tuning parameters, such as kernels, bandwidths or lags. Also, based on the sequence of IM-OLS residuals, we propose some new test statistics based on different measures of excessive fluctuation for testing the null hypothesis of cointegration against the alternative of no cointegration. For these test statistics we derive their asymptotic null and alternative distributions, and study their finite sample performance through a local-to-unity approach to the null of cointegration.

Keywords and phrases: cointegration, asymptotically efficient estimation, OLS, FM-OLS, IM-OLS, trending integrated regressors

JEL Classification: C13, C32

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1. Introduction

Cointegration analysis is widely used in empirical macroeconomics and finance, and includes both the estimation of cointegrating relationships and hypothesis testing, and also testing the hypothesis of cointegration among nonstationary variables. In the econometric literature there are many contributions in these two topics, some of which deals with these two questions simultaneously. Given the usual linear specification of a potentially cointegrating regression, a first candidate for estimation is the method of ordinary least squares (OLS), that determines superconsistent estimates of the regression parameters under cointegration. However, with endogenous regressors the limiting distribution of the OLS estimator is contaminated by a number of nuisance parameters, also known as second order bias terms, which renders inference problematic. Consequently, there has been proposed several modifications to OLS to makes standard asymptotic inference feasible but at the cost of introducing the choice of several tuning parameters and functions. These methods include the fully modified OLS (FM-OLS) approach of Phillips and Hansen (1990), the canonical cointegrating regression (CCR) by Park (1992), and the dynamic OLS (DOLS) approach of Phillips and Loretan (1991), Saikkonen (1991) and Stock and Watson (1993).

This paper deals with the analysis of a new asymptotically efficient estimation method of a linear cointegrating regression recently proposed by Vogelsang and Wagner (2011) that does not require any additional choice more than the initial standard assumptions on the model specification, making it a very appealing alternative.

This methods, which is called the integrated modified OLS (IM-OLS) estimator, works under a simple transformation of the model variables that asymptotically produces the same correction effect as the commonly used estimation methods cited above.

An important issue, which is often is not taken into account and that can substantially affect the performance and properties of these estimation procedures, is the nature and structure of the deterministic component, if any, of the generating mechanism of the model variables and its relation with the deterministic component, if is considered, in the specification of the cointegrating regression. Following the work by Hansen (1992), we generalize its formulation by allowing for deterministically trending integrated regressors with a possibly different structure for their deterministic components and propose a simple rule for a proper specification of the deterministic trend function in the cointegrating regression that simultaneously correct for their effects.

Given the particular transformation of the model variables required for performing the asymptotically efficient IM-OLS estimation, we show that a proper accommodation of these components must be based on a previous transformation of the model variables, in particular the OLS detrending. With these corrected observations we perform the IM-OLS estimation of the cointegrating regression and derive the limiting distributions of the resulting estimates and residuals both under the assumption of cointegration and no cointegration.

Based on these new asymptotically efficient estimators of the vector parameters in the cointegrating regression, we study the building of some simple statistics for testing the null hypothesis of cointegration by using different measures of excessive fluctuation in the IM-OLS residual sequence that cannot be compatible with the stationarity assumption of the error sequence. These test statistics are based on the statistics proposed by Shin (1994), Xiao and Phillips (2002) and Wu and Xiao (2008) with the same objective as ours, and that use two basic measures of excessive fluctuations, the Cramér-von Mises (CvM) and Kolmogorov-Smirnov (KS) metrics. We derive their limiting null and alternative distributions and evaluate their power behavior in finite-samples through a simulation experiment.

2. The model, OLS and FM-OLS estimation of the linear cointegrating regression with trending regressors

We assume that the variables of interest, the scalar Y_t and the k-dimensional vector $\mathbf{X}_{k,t} = (X_{1,t},...,X_{k,t})'$, come from the following data generating process (DGP)

$$\begin{pmatrix} Y_t \\ \mathbf{X}_{k,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}'_{0,p}\boldsymbol{\tau}_{p,t} \\ \mathbf{A}_{k,p}\boldsymbol{\tau}_{p,t} \end{pmatrix} + \begin{pmatrix} \eta_{0,t} \\ \boldsymbol{\eta}_{k,t} \end{pmatrix} \quad t = 1,...,n \tag{2.1}$$

where $\boldsymbol{\eta}_t = (\eta_{0,t}, \boldsymbol{\eta}'_{k,t})'$ is the stochastic trend component that satisfy the first order recurrence relation

$$\boldsymbol{\eta}_t = \boldsymbol{\eta}_{t-1} + \boldsymbol{\varepsilon}_t$$

with $\boldsymbol{\varepsilon}_t = (\varepsilon_{0,t}, \boldsymbol{\varepsilon}'_{k,t})'$ a k+1 vector zero mean sequence of error processes. Also, we consider the general case where both Y_t and each element of the k vector $\mathbf{X}_{k,t} = (X_{1,t},...,X_{k,t})'$ contains a deterministic trend component given by a polynomial trend function of an arbitrary order $p_i \geq 0$, $i = 0, 1, \ldots, k$, that is $d_{i,t} = \boldsymbol{\alpha}'_{i,p_i}\boldsymbol{\tau}_{p_i,t}$, with $\boldsymbol{\alpha}_{i,p_i} = (\alpha_{i,0}, \alpha_{i,1},..., \alpha_{i,p_i})'$, and $\boldsymbol{\tau}_{p_i,t} = (1, t, ..., t^{p_i})'$. To make this assumption compatible with the standard formulation in (2.1) where all the deterministic trend components appears as if it were of the same type and order, we have to write

$$\boldsymbol{\alpha}'_{i,p_i}\boldsymbol{\tau}_{p_i,t} = (\boldsymbol{\alpha}'_{i,p_i} : \mathbf{0}'_{p-p_i}) \begin{pmatrix} \boldsymbol{\tau}_{p_i,t} \\ \boldsymbol{\tau}_{p-p_i,t} \end{pmatrix} = \boldsymbol{\alpha}'_{i,p}\boldsymbol{\tau}_{p,t}, \qquad i = 0, 1, \ldots, k \tag{2.2}$$

with $p = \max(p_0, p_1, ..., p_k)$ and $\mathbf{0}_{p-p_i}$ a $(p-p_i)\times 1$ vector of zeroes, so that

$$\mathbf{A}_{k,p}\boldsymbol{\tau}_{p,t} = \begin{pmatrix} \boldsymbol{\alpha}'_{1,p_1}\boldsymbol{\tau}_{p_1,t} \\ \vdots \\ \boldsymbol{\alpha}'_{k,p_k}\boldsymbol{\tau}_{p_k,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}'_{1,p} \\ \vdots \\ \boldsymbol{\alpha}'_{k,p} \end{pmatrix} \boldsymbol{\tau}_{p,t}$$

With this formulation, we introduce the static potentially cointegrating regression equation between the unobserved stochastic trend components of the elements in $\mathbf{Z}_t$ as

$$\eta_{0,t} = \boldsymbol{\eta}'_{k,t}\boldsymbol{\beta}_k + u_t$$

which gives

$$Y_t = \boldsymbol{\alpha}_p' \boldsymbol{\tau}_{p,t} + \boldsymbol{\beta}_k' \mathbf{X}_{k,t} + u_t \quad t = 1,\dots,n \tag{2.3}$$

with $\boldsymbol{\alpha}_p = \boldsymbol{\alpha}_{0,p} - \mathbf{A}_{k,p}' \boldsymbol{\beta}_k$. Associated to the deterministic component we introduce the polynomial order trend and sample size dependent scaling matrix $\boldsymbol{\Gamma}_{p,n}$, given by $\boldsymbol{\Gamma}_{p,n} = diag(1, n^{-1}, \dots, n^{-p})$, which determines that $\boldsymbol{\tau}_{p,tn} = \boldsymbol{\Gamma}_{p,n} \boldsymbol{\tau}_{p,t} \to \boldsymbol{\tau}_p(r) = (1, r, \dots, r^p)'$ uniformly over $r \in [0,1]$ as $n \to \infty$. Also we have that $n^{-1} \sum_{t=1}^{[nr]} \boldsymbol{\tau}_{p,tn} \to \int_0^r \boldsymbol{\tau}_p(s) ds$, and $n^{-1} \sum_{t=1}^{n} \boldsymbol{\tau}_{p,tn} \boldsymbol{\tau}_{p,tn}' = n^{-1} \mathbf{Q}_{n,pp} = \overline{\mathbf{Q}}_{n,pp} \to \mathbf{Q}_{pp} = \int_0^1 \boldsymbol{\tau}_p(s) \boldsymbol{\tau}_p(s)' ds < \infty$. In order to complete the specification of our data generating process we next introduce a quite general and common assumption on the error terms involved in (2.3).

Assumption 2.1. We assume that the error term in the cointegrating regression u_t satisfy the first-order recurrence relation $u_t = \alpha u_{t-1} + \upsilon_t$, with $|\alpha| \leq 1$, where the zero mean (k+1)-dimensional error sequence $\boldsymbol{\xi}_t = (\upsilon_t, \boldsymbol{\varepsilon}_{k,t}')'$ verify any of the existing conditions that guarantee the validity of the functional central limit theorem (FCLT) approximation of the form

$$n^{-1/2} \sum_{t=1}^{[nr]} \begin{pmatrix} \upsilon_t \\ \boldsymbol{\varepsilon}_{k,t} \end{pmatrix} \Rightarrow \mathbf{B}(r) = \begin{pmatrix} B_\upsilon(r) \\ \mathbf{B}_k(r) \end{pmatrix} = \mathbf{BM}(\boldsymbol{\Omega}) = \boldsymbol{\Omega}^{1/2} \mathbf{W}(r) \quad 0 \leq r \leq 1$$

with $\mathbf{W}(r) = (W_\upsilon(r), \mathbf{W}_k'(r))'$ a k+1-dimensional standard Brownian motion, and $\boldsymbol{\Omega}$ the covariance matrix of $\mathbf{B}(r)$, which is assumed to be positive definite and that can also be interpreted as the long-run covariance matrix of the vector error sequence $\boldsymbol{\xi}_t$, that is $\boldsymbol{\Omega} = E[\boldsymbol{\xi}_t \boldsymbol{\xi}_t'] + \sum_{j=1}^{\infty} (E[\boldsymbol{\xi}_t \boldsymbol{\xi}_{t-j}'] + E[\boldsymbol{\xi}_{t-j} \boldsymbol{\xi}_t'])$, which can be decomposed as $\boldsymbol{\Omega} = \boldsymbol{\Delta} + \boldsymbol{\Lambda}'$, with $\boldsymbol{\Delta} = \boldsymbol{\Sigma} + \boldsymbol{\Lambda}$ the one-sided long-run covariance matrix, where $\boldsymbol{\Sigma} = E[\boldsymbol{\xi}_t \boldsymbol{\xi}_t']$, and $\boldsymbol{\Lambda} = \sum_{j=1}^{\infty} E[\boldsymbol{\xi}_t \boldsymbol{\xi}_{t-j}']$.

The assumption of positive definiteness of $\boldsymbol{\Omega}$ excludes cointegration among the k

integrated regressors $\mathbf{X}_{k,t}$ (subcointegration) with $\mathbf{B}_k(r) = \mathbf{BM}(\mathbf{\Omega}_{k,k})$, $\mathbf{\Omega}_{k,k} > 0$. Given the upper triangular Cholesky decomposition of the matrix $\mathbf{\Omega}$, we then have that $B_\upsilon(r) = B_{\upsilon.k}(r) + \boldsymbol{\gamma}'_{k\upsilon}\mathbf{B}_k(r)$, with $B_{\upsilon.k}(r) = \omega_{\upsilon.k}W_\upsilon(r)$, and $\mathbf{B}_k(r) = \mathbf{\Omega}^{1/2}_{k,k}\mathbf{W}_k(r)$, where $\boldsymbol{\gamma}_{k\upsilon} = \mathbf{\Omega}^{-1}_{k,k}\boldsymbol{\omega}'_{\upsilon,k}$ and $\omega^2_{\upsilon.k} = E[B_{\upsilon.k}(r)^2] = E[B_{\upsilon.k}(r)B_\upsilon(r)] = \omega^2_\upsilon - \boldsymbol{\omega}_{\upsilon,k}\mathbf{\Omega}^{-1}_{k,k}\boldsymbol{\omega}'_{\upsilon,k}$ is the conditional variance of $B_\upsilon(r)$ given $\mathbf{B}_k(r)$, which gives $E[\mathbf{B}_k(r)B_{\upsilon.k}(r)] = \mathbf{0}_k$.

For the initial values $\boldsymbol{\eta}_{k,0}$ and u_0, we introduce the very general conditions $\boldsymbol{\eta}_{k,0} = o_p(n^{1/2})$, and $u_0 = o_p(n^{1/2})$, which include the particular case of constant finite values. In the case of a stationary error term u_t, with $|\alpha| < 1$, we then have that $n^{-1/2}\sum_{t=1}^{[nr]} u_t \Rightarrow B_u(r) = (1-\alpha)^{-1}B_\upsilon(r)$, with $B_u(r) = B_{u.k}(r) + \boldsymbol{\gamma}'_k\mathbf{B}_k(r)$, $\boldsymbol{\gamma}_k = \mathbf{\Omega}^{-1}_{kk}\boldsymbol{\omega}_{ku}$, $E[B_u(r)^2] = \omega^2_u = (1-\alpha)^{-2}\omega^2_\upsilon$, $E[B_{u.k}(r)^2] = \omega^2_{u.k} = \omega^2_u - \boldsymbol{\gamma}'_k\mathbf{\Omega}_{kk}\boldsymbol{\gamma}_k$, and $E[\mathbf{B}_k(r)B_u(r)] = \boldsymbol{\omega}_{ku} = (1-\alpha)^{-1}\boldsymbol{\omega}_{k\upsilon}$, while that in the case of nonstationarity, that is when $\alpha = 1$, then $n^{-1/2}u_{[nr]} \Rightarrow B_u(r) = B_\upsilon(r)$, with $\omega^2_u = \omega^2_\upsilon$. With these results then we have

$$n^{-(1-v)}U_{[nr]} = n^{-(1-v)}\sum_{t=1}^{[nr]} u_t \Rightarrow J_u(r) = \begin{cases} B_u(r) & v = 1/2 \\ \int_0^r B_u(s)ds & v = -1/2 \end{cases}$$

with $v = 1/2$ and $v = -1/2$ indicating, respectively, the stationary and nonstationary cases. Given the specification of the linear static cointegrating regression equation (2.3) the standard approach to estimating the vector parameters $\boldsymbol{\alpha}_p, \boldsymbol{\beta}_k$ consists in the use of the OLS estimation which gives

$$\begin{pmatrix} \hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p \\ \hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k \end{pmatrix} = \left(\sum_{t=1}^{n} \begin{pmatrix} \boldsymbol{\tau}_{p,t} \\ \mathbf{X}_{k,t} \end{pmatrix} (\boldsymbol{\tau}'_{p,t}, \mathbf{X}'_{k,t}) \right)^{-1} \sum_{t=1}^{n} \begin{pmatrix} \boldsymbol{\tau}_{p,t} \\ \mathbf{X}_{k,t} \end{pmatrix} u_t$$

Taking into account the structure for the deterministic and stochastic trend components of the observed processes Y_t and $\mathbf{X}_{k,t}$ in (2.1), we can write

$$
\begin{aligned}
\begin{pmatrix} \boldsymbol{\tau}_{p,t} \\ \mathbf{X}_{k,t} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\Gamma}_{p,n}^{-1}\boldsymbol{\tau}_{p,tn} \\ \mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1}\boldsymbol{\tau}_{p,tn} + \boldsymbol{\eta}_{k,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Gamma}_{p,n}^{-1} & \mathbf{0}_{p+1,k} \\ \mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1} & \sqrt{n}\mathbf{I}_{k,k} \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} \\
&= \mathbf{W}_n \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} = \mathbf{W}_n \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ n^{-1/2}\boldsymbol{\eta}_{k,tn} \end{pmatrix}
\end{aligned} \tag{2.4}
$$

so that

$$
\begin{aligned}
\begin{pmatrix} \hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p \\ \hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k \end{pmatrix} &= n^{-v}(\mathbf{W}_n')^{-1} \left((1/n)\sum_{t=1}^{n} \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} (\boldsymbol{\tau}_{p,tn}', n^{-1/2}\boldsymbol{\eta}_{k,t}') \right)^{-1} \\
&\quad \times n^{-(1-v)} \sum_{t=1}^{n} \left\{ \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} u_t \right\}
\end{aligned}
$$

with the power v taking values $\pm 1/2$ depending on the stochastic properties of the error sequence u_t, and determining the order of consistency of the OLS estimates, that is

$$
\begin{aligned}
&\begin{pmatrix} n^{v}\boldsymbol{\Gamma}_{p,n}^{-1}[(\hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p) + \mathbf{A}_{k,p}'(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)] \\ n^{1/2+v}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \end{pmatrix} \\
&= \left((1/n)\sum_{t=1}^{n} \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} (\boldsymbol{\tau}_{p,tn}', n^{-1/2}\boldsymbol{\eta}_{k,t}') \right)^{-1} \left\{ n^{-(1-v)} \sum_{t=1}^{n} \begin{pmatrix} \boldsymbol{\tau}_{p,tn} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} u_t \right\}
\end{aligned} \tag{2.5}
$$

The usual result in this context is as in (2.5) but with $\mathbf{A}_{k,p} = \mathbf{0}_{k,p+1}$, which corresponds to the case where the integrated regressors have no deterministic component and, in out formulation, the deterministic term that appears in the cointegrating equation corresponds to the one included in Y_t. Hansen (1992) has studied a similar situation, but assuming that $Y_t = \eta_{0,t}$ with $d_{0,t} = \boldsymbol{\alpha}_{0,p_0}' \boldsymbol{\tau}_{p_0,t} = 0$, and $p_i = m$, $i = 1, \ldots, k$, with $\boldsymbol{\tau}_{m,t} = (t^{p_1}, t^{p_2}, \ldots, t^{p_m})'$, $1 \le p_1 < \ldots < p_m$, and scaling matrix $\boldsymbol{\Gamma}_{m,n} = diag(n^{-p_1}, n^{-p_2}, \ldots, n^{-p_m})$ (see Theorem 1(a, b), p.93). The main differences with our approach are the no inclusion of a constant term and the inclusion of a rank condition on the coefficient matrix $\mathbf{A}_{k,m}$, particularly, rank($\mathbf{A}_{k,m}$) = $m \le k$. Then, from (2.1) we have that

$$
\begin{aligned}
\boldsymbol{\Gamma}_{m,n}[(\mathbf{A}_{k,m}'\mathbf{A}_{k,m})^{-1}\mathbf{A}_{k,m}']\mathbf{X}_{k,[nr]} &= \boldsymbol{\tau}_{m,[nr]n} + \sqrt{n}\boldsymbol{\Gamma}_{m,n}[(\mathbf{A}_{k,m}'\mathbf{A}_{k,m})^{-1}\mathbf{A}_{k,m}'](n^{-1/2}\boldsymbol{\eta}_{k,[nr]}) \\
&= \boldsymbol{\tau}_{m,[nr]n} + O_p(n^{-(p_1-1/2)}) \Rightarrow \boldsymbol{\tau}_m(r)
\end{aligned}
$$

which allows the possibility to develop a sequence of weights which yield a nondegenerate design limiting matrix when estimating (2.3) by OLS under the restriction $\boldsymbol{\alpha}_m = \mathbf{0}_m$. However, as can see from the previous result, this only yields consistent results when $p_1 \geq 1$, and there is no constant term in the regression neither in the polynomial trend function.[1] Under the assumption of cointegration (v = 1/2), then the limiting distribution of the last term in (2.5) is given by

$$n^{-(1-v)}\sum_{t=1}^{[nr]}\begin{pmatrix}\boldsymbol{\tau}_{p,tn}\\ n^{-1/2}\boldsymbol{\eta}_{k,t}\end{pmatrix}u_t \Rightarrow \int_0^r\begin{pmatrix}\boldsymbol{\tau}_p(s)\\ \mathbf{B}_k(s)\end{pmatrix}dB_u(s)+\begin{pmatrix}\mathbf{0}_{p+1}\\ r\boldsymbol{\Delta}_{k,u}\end{pmatrix}$$
$$=\left\{\int_0^r\begin{pmatrix}\boldsymbol{\tau}_p(s)\\ \mathbf{B}_k(s)\end{pmatrix}dB_{u.k}(s)+\int_0^r\begin{pmatrix}\boldsymbol{\tau}_p(s)\\ \mathbf{B}_k(s)\end{pmatrix}d\mathbf{B}_k(s)'\boldsymbol{\gamma}_k\right\}+\begin{pmatrix}\mathbf{0}_{p+1}\\ r\boldsymbol{\Delta}_{k,u}\end{pmatrix}$$

with $\boldsymbol{\Delta}_{k,u}=\sum_{j=0}^{\infty}E[\boldsymbol{\varepsilon}_{k,t-j}u_t]$ given by the probability limit of $\boldsymbol{\Delta}_{n,ku}=n^{-1}\sum_{t=1}^{n}E[\boldsymbol{\eta}_{k,t}u_t]$.[2] This limiting distribution contains the second-order bias due to the correlation between $B_u(\cdot)$ and $\mathbf{B}_k(\cdot)$ (endogeneity of the stochastic trend components of the regressors), and the non-centrality bias that comes from the fact that the regression errors are serially correlated through the parameter $\boldsymbol{\Delta}_{k,u}$. For the first term above we have that

$$\int_0^1\mathbf{B}_k(s)dB_{u.k}(s)=\omega_{u.k}\boldsymbol{\Omega}_{kk}^{1/2}\int_0^1\mathbf{W}_k(s)dW_{u.k}(s)$$

where, given that $\mathbf{B}_k(r)$ and $B_{u.k}(r)$ are independent, conditioning on $\mathbf{B}_k(r)$ (or $\mathbf{W}_k(r)$) can be used to show that this term is a zero mean Gaussian mixture of the form

$$\int_0^1\mathbf{W}_k(s)dW_{u.k}(s)=\int_{\mathbf{G}_{k,k}>0}N(\mathbf{0}_k,\mathbf{G}_{k,k}^{-1})dP(\mathbf{G}_{k,k}),\quad \mathbf{G}_{k,k}=\left(\int_0^1\mathbf{W}_k(s)\mathbf{W}_k(s)'\right)^{-1}$$

The second term in the expression between brackets is a matrix unit root distribution, arising from the k stochastic trends in $\mathbf{X}_{k,t}$, which is cancelled under strict exogeneity of

[1] See also Hassler (2001) for a related study in the case where the specification of the cointegrating regression equation does not include any deterministic term but the integrated regressors $\mathbf{X}_{k,t}$ contain a constant term.

[2] The result $r\boldsymbol{\Delta}_{k,u}$ is obtained by writing $\boldsymbol{\Delta}_{n,ku}(r)=n^{-1}\sum_{t=1}^{[nr]}E[\boldsymbol{\eta}_{k,t}u_t]=\frac{[nr]}{n}([nr])^{-1}\sum_{t=1}^{[nr]}E[\boldsymbol{\eta}_{k,t}u_t]$, so that

$$\boldsymbol{\Delta}_{n,ku}(r)=\tfrac{[nr]}{n}[([nr])^{-1}\sum_{t=1}^{[nr]}E[\boldsymbol{\eta}_{k,0}u_t]+\sum_{j=0}^{[nr]-1}(([nr])^{-1}\sum_{t=j+1}^{[nr]}E[\boldsymbol{\varepsilon}_{k,t-j}u_t])]$$

and the use of the initial condition $\boldsymbol{\eta}_{k,0}$, and Assumption 2.1 on the properties of the error terms.

the regressors, that is when $\boldsymbol{\omega}_{ku} = \mathbf{0}_k$. Using now (2.5) and the decomposition for $\mathbf{X}_{k,t}$ in (2.1), we have that the sequence of OLS residuals is given by

$$\begin{aligned}\hat{u}_{t,p}(k) &= u_t - \boldsymbol{\tau}'_{p,t}(\hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p) - \mathbf{X}'_{k,t}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ &= u_t - \boldsymbol{\tau}'_{p,tn}\{\boldsymbol{\Gamma}^{-1}_{p,n}[(\hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p) + \mathbf{A}'_{k,p}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)]\} - (n^{-1/2}\boldsymbol{\eta}'_{k,t})[\sqrt{n}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)]\end{aligned}$$

where the first element component in (2.5) can be written as

$$\begin{aligned}&\boldsymbol{\Gamma}^{-1}_{p,n}[(\hat{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p) + \mathbf{A}'_{k,p}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)] \\ &\qquad = \mathbf{Q}^{-1}_{n,pp}\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}u_t - \mathbf{Q}^{-1}_{n,pp}\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}(n^{-1/2}\boldsymbol{\eta}'_{k,t})[\sqrt{n}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)]\end{aligned}$$

which gives

$$\begin{aligned}\hat{u}_{t,p}(k) &= u_t - \boldsymbol{\tau}'_{p,tn}\mathbf{Q}^{-1}_{n,pp}\sum_{j=1}^{n}\boldsymbol{\tau}_{p,jn}u_j \\ &\quad - n^{-1/2}\left\{\boldsymbol{\eta}'_{k,t} - \boldsymbol{\tau}'_{p,tn}\mathbf{Q}^{-1}_{n,pp}\sum_{j=1}^{n}\boldsymbol{\tau}_{p,jn}\boldsymbol{\eta}'_{k,j}\right\}[\sqrt{n}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)] \\ &= u_{t,p} - n^{-(1/2+v)}\boldsymbol{\eta}'_{kt,p}[n^{1/2+v}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)]\end{aligned} \qquad (2.7)$$

so that the OLS residuals are free of the trend parameters, and are decomposed in terms of the detrended versions of u_t and $\boldsymbol{\eta}_{k,t}$ as defined in (2.7). These OLS residuals can be used as the basis for building some simple statistics for testing the null hypothesis of cointegration against the alternative of no cointegration, given that $\hat{u}_{t,p}(k) = O_p(1)$ when $v = 1/2$, and $\hat{u}_{t,p}(k) = O_p(n^{1/2})$ when $v = -1/2$. This difference in behavior under the null and the alternative can be exploited by searching for excessive fluctuations in the sequence of scaled partial sum of residuals $\hat{B}_{[nr],p}(k) = n^{-1/2}\sum_{t=1}^{[nr]}\hat{u}_{t,p}(k)$ through several global measures, such as a Cramér-von Mises (CvM) measure of fluctuation as in Shin (1994), or a Kolmogorov-Smirnov (KS) measure as in Xiao and Phillips (2002), and Wu and Xiao (2008).[3] The CvM-type test by Shin (1994) is based on a global measure of fluctuation given by $S_{n,p}(k) = (1/n)\sum_{t=1}^{n}(\hat{B}_{t,p}(k))^2$, while that the KS-type test statistic

[3] The test statistic proposed by Shin (1994) is the generalization of the KPSS statistic for the null of stationarity by Kwiatkowski et.al. (1992), while the test statistics considered in Xiao and Phillips (2002), and Wu and Xiao (2008) are the generalizations of the KS test statistic formulated by Xiao (2001).

proposed by Wu and Xiao (2008) is based on the recursive centered measure of maximum fluctuation $R_{n,p}(k)=\max_{t=1,\ldots,n}|\hat{B}_{t,p}(k)-(t/n)\hat{B}_{n,p}(k)|$. Xiao and Phillips (2002) considers a no centered version of this test statistic given by $CS_{n,p}(k)=\max_{t=1,\ldots,n}|\hat{B}_{t,p}(k)|$, which is the same as $R_{n,p}(k)$ when based on OLS residuals and the deterministic component contains a constant term. The main problem with this approach is that, unless corrected, the null distribution of all these test statistics are plagued of nuisance parameters due to endogeneity of regressors and the serial correlation in the error terms that cannot be removed by simple scaling methods. There exist some different methods, which are known as asymptotically efficient estimation methods, to remove these parameters and that differ in the treatment of each source of bias. Among the existing estimation methods, the three most commonly used are the Dynamic OLS (DOLS) estimator proposed by Phillips and Loretan (1991), Saikkonen (1991) and Stock and Watson (1993), the Canonical Cointegrating Regression (CCR) estimator by Park (1992), and the Fully-Modified OLS (FM-OLS) estimator by Phillips and Hansen (1990). These three estimators are asymptotically equivalent and, as was proved by Saikkonen (1991), efficient. The corrected test statistic proposed by Shin (1994) makes use of the DOLS residuals, while that the test statistics considered in Xiao and Phillips (2002) and Wu and Xiao (2008) are based on FM-OLS residuals. For a recent review and comparison of these three alternative estimation methods see, e.g., Kurozumi and Hayakawa (2009), and references therein. In order to establish the basis for our proposal in the next section we consider an alternative to the cointegrating regression equation (2.3). By applying the partitioned OLS estimation to the regression equation (2.3) with respect to the trend parameters, we have that this model can also be written as

$$\hat{Y}_{t,p}=\boldsymbol{\beta}_k'\hat{\mathbf{X}}_{kt,p}+u_{t,p},\quad t=1,\ldots,n \tag{2.8}$$

where $\hat{Y}_{t,p} = Y_t - \boldsymbol{\tau}'_{p,tn}\mathbf{Q}^{-1}_{n,pp}\sum_{j=1}^{n}\boldsymbol{\tau}_{p,jn}Y_j$, $\hat{\mathbf{X}}_{kt,p} = \mathbf{X}_{k,t} - \sum_{j=1}^{n}\mathbf{X}_{k,j}\boldsymbol{\tau}'_{p,jn}\mathbf{Q}^{-1}_{n,pp}\boldsymbol{\tau}_{p,tn}$, and $u_{t,p} = u_t - \boldsymbol{\tau}'_{p,tn}\mathbf{Q}^{-1}_{n,pp}\sum_{j=1}^{n}\boldsymbol{\tau}_{p,jn}u_j$ denote the detrended observations of the model variables obtained by OLS fitting of their original observations to a pth-order polynomial trend function, where p is chosen according to the rule $p \geq \max(p_0, p_1, ..., p_k)$ in the case where the polynomial trend functions in Y_t and each component of $\mathbf{X}_{k,t}$ differ in their orders. The next Proposition 2.1 determines the effectiveness of this procedure to make the estimation results invariant to the trend parameters in (2.1).

Proposition 2.1. *Given (2.1)-(2.2), when considering the OLS detrending of Y_t and $\mathbf{X}_{k,t}$ by fitting a polynomial trend function of order $p = \max(p_0, p_1, ..., p_k)$ to each of these variables, then we have that*

$$\hat{Y}_{t,p} = \eta_{0t,p} = \eta_{0,t} - \boldsymbol{\tau}'_{p,tn}\mathbf{Q}^{-1}_{n,pp}\sum_{j=1}^{n}\boldsymbol{\tau}_{p,jn}\eta_{0,j}$$

and

$$\hat{\mathbf{X}}_{kt,p} = \boldsymbol{\eta}_{kt,p} = \boldsymbol{\eta}_{k,t} - \sum_{j=1}^{n}\boldsymbol{\eta}_{k,j}\boldsymbol{\tau}'_{p,jn}\mathbf{Q}^{-1}_{n,pp}\boldsymbol{\tau}_{p,tn} = (\eta_{1t,p},...,\eta_{kt,p})'$$

where $\eta_{0t,p}$ and $\boldsymbol{\eta}_{kt,p}$ are generalized detrended transformations of $\eta_{0,t}$ and $\boldsymbol{\eta}_{k,t}$, with

$$n^{-1/2}\boldsymbol{\eta}_{k[nr],p} \Rightarrow \mathbf{B}_{k,p}(r) = \mathbf{B}_k(r) - \int_0^1 \mathbf{B}_k(s)\boldsymbol{\tau}_p(s)'ds\,\mathbf{Q}^{-1}_{pp}\boldsymbol{\tau}_p(r) \quad (2.9)$$

a (p+1)-order detrended transformation of $\mathbf{B}_k(r)$. According to Lemma A.2 in Phillips and Hansen (1990), $\mathbf{B}_{k,p}(r) = \mathbf{BM}(\boldsymbol{\Omega}_{k,k}\cdot v_p(r))$ is a full rank Gaussian processes, with $v_p(r)$ a scalar function of r and $\boldsymbol{\tau}_p(\cdot)$.

Proof. See Appendix A.

By OLS estimation of the cointegrating vector component $\boldsymbol{\beta}_k$ in (2.8) we have

$$n^{(1/2+v)}(\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) = \left(\sum_{t=1}^{n}\hat{\mathbf{X}}_{kt,p}\hat{\mathbf{X}}'_{kt,p}\right)^{-1}\sum_{t=1}^{n}\hat{\mathbf{X}}_{kt,p}u_{t,p}$$
$$= \left((1/n)\sum_{t=1}^{n}(n^{-1/2}\boldsymbol{\eta}_{kt,p})(n^{-1/2}\boldsymbol{\eta}'_{kt,p})\right)^{-1} n^{-(1-v)}\sum_{t=1}^{n}(n^{-1/2}\boldsymbol{\eta}_{kt,p})u_{t,p}$$

which gives the same limiting results as before under cointegration, that is with $v = 1/2$, to that $n^{-1/2}\sum_{j=1}^{n}\boldsymbol{\tau}_{p,jn}u_j \Rightarrow \int_0^1\boldsymbol{\tau}_p(s)dB_u(s)$, and thus $u_{t,p} = u_t + O_p(n^{-1/2})$. In order to complete the above results, we next consider the relationship between the FM-OLS and OLS estimators of $\boldsymbol{\alpha}_p$ and $\boldsymbol{\beta}_k$ in (2.3), which is given by

$$\begin{pmatrix}\hat{\boldsymbol{\alpha}}^{+}_{p,n}\\ \hat{\boldsymbol{\beta}}^{+}_{k,n}\end{pmatrix}=\left(\sum_{t=1}^{n}\begin{pmatrix}\boldsymbol{\tau}_{p,t}\\ \mathbf{X}_{k,t}\end{pmatrix}(\boldsymbol{\tau}'_{p,t},\mathbf{X}'_{k,t})\right)^{-1}\left\{\sum_{t=1}^{n}\begin{pmatrix}\boldsymbol{\tau}_{p,t}\\ \mathbf{X}_{k,t}\end{pmatrix}Y_t^{+}-\begin{pmatrix}\mathbf{0}_{p+1}\\ n\boldsymbol{\Delta}^{+}_{ku}\end{pmatrix}\right\} \tag{2.10}$$

where $Y_t^{+}=Y_t-\Delta\mathbf{X}'_{k,t}\boldsymbol{\gamma}_k$, $\boldsymbol{\Delta}^{+}_{ku}=\boldsymbol{\Delta}_{ku}-\boldsymbol{\Delta}_{kk}\boldsymbol{\gamma}_k$, and $\boldsymbol{\gamma}_k=\boldsymbol{\Omega}^{-1}_{kk}\boldsymbol{\omega}_{ku}$.[4] Again, as in (2.1)-(2.2), the key is to consider the following general decomposition for $\mathbf{Z}_{k,t}=\Delta\mathbf{X}_{k,t}$

$$\begin{aligned}\Delta\mathbf{X}_{k,t}&=\mathbf{A}_{k,p}\Delta\boldsymbol{\tau}_{p,t}+\Delta\boldsymbol{\eta}_{k,t}=\boldsymbol{\Phi}_{k,p-1}\boldsymbol{\tau}_{p-1,t}+\boldsymbol{\varepsilon}_{k,t}\\ &=(\boldsymbol{\Phi}_{k,p-1}:\mathbf{0}_k)\begin{pmatrix}\boldsymbol{\tau}_{p-1,t}\\ \tau_{p,t}\end{pmatrix}+\boldsymbol{\varepsilon}_{k,t}=\boldsymbol{\Phi}_{k,p}\boldsymbol{\Gamma}^{-1}_{p,n}\boldsymbol{\tau}_{p,tn}+\boldsymbol{\varepsilon}_{k,t}\end{aligned} \tag{2.11}$$

with the matrix of trend coefficients $\boldsymbol{\Phi}_{k,p-1}$ given by a linear combination of the corresponding elements of $\mathbf{A}_{k,p}$.

Proposition 2.2. *Given (2.1)-(2.4), and the FM-OLS estimator of (2.3) in (2.10), then we have that*

(a)

$$\begin{pmatrix}\boldsymbol{\Gamma}^{-1}_{p,n}[\hat{\boldsymbol{\alpha}}^{+}_{p,n}+\mathbf{A}'_{k,p}\hat{\boldsymbol{\beta}}^{+}_{k,n}]\\ \sqrt{n}\hat{\boldsymbol{\beta}}^{+}_{k,n}\end{pmatrix}=\begin{pmatrix}\boldsymbol{\Gamma}^{-1}_{p,n}[\hat{\boldsymbol{\alpha}}_{p,n}+\mathbf{A}'_{k,p}\hat{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\Phi}'_{k,p}\boldsymbol{\gamma}_k]\\ \sqrt{n}\hat{\boldsymbol{\beta}}_{k,n}\end{pmatrix}$$
$$-\begin{pmatrix}\mathbf{M}^{-1}_{n,pp}\left\{\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{k,t}\right\}\boldsymbol{\gamma}_k-\sqrt{n}\mathbf{M}^{-1}_{n,pp}\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\boldsymbol{\Delta}^{+}_{ku}\\ \mathbf{M}^{-1}_{n,kk}\left\{\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}'_{n,pk}\mathbf{Q}^{-1}_{n,pp}\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}\right\}\boldsymbol{\gamma}_k+\sqrt{n}\mathbf{M}^{-1}_{n,kk}\boldsymbol{\Delta}^{+}_{ku}\end{pmatrix} \tag{2.12}$$

with FM-OLS residuals, such that

$$\begin{aligned}\text{(b)}\ \hat{u}^{+}_{t,p}(k)&=\hat{u}_{t,p}(k)-\boldsymbol{\varepsilon}'_{kt,p}\boldsymbol{\gamma}_k\\ &+(1/\sqrt{n})(n^{-1/2}\boldsymbol{\eta}'_{kt,p})\bar{\mathbf{M}}^{-1}_{n,kk}\left((1/\sqrt{n})\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{kt,p}\cdot\boldsymbol{\gamma}_k+\boldsymbol{\Delta}^{+}_{ku}\right)\end{aligned} \tag{2.13}$$

where $\mathbf{M}_{n,pp}=\mathbf{Q}_{n,pp}-\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\mathbf{Q}'_{n,pk}$, $\mathbf{Q}_{n,pk}=\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\eta}'_{k,t}$, $\mathbf{Q}_{n,kk}=\sum_{t=1}^{n}\boldsymbol{\eta}_{k,t}\boldsymbol{\eta}'_{k,t}$, *and* $\bar{\mathbf{M}}_{n,kk}=(1/n)\mathbf{M}_{n,kk}$, *with* $\mathbf{M}_{n,kk}=\sum_{t=1}^{n}\boldsymbol{\eta}_{kt,p}\boldsymbol{\eta}'_{kt,p}$, *and* $\boldsymbol{\varepsilon}_{kt,p}=\boldsymbol{\varepsilon}_{k,t}-\sum_{j=1}^{n}\boldsymbol{\varepsilon}_{k,j}\boldsymbol{\tau}'_{p,jn}\mathbf{Q}^{-1}_{n,pp}\boldsymbol{\tau}_{p,tn}$.

Proof. See Appendix B.

Remark 2.1. The FM-OLS estimator in (2.10), as well as the results in (2.12) and (2.13), is not feasible since it is defined in terms of the unknown quantities $\boldsymbol{\gamma}_k$ and $\boldsymbol{\Delta}^{+}_{ku}=\boldsymbol{\Delta}_{ku}-\boldsymbol{\Delta}_{kk}\boldsymbol{\gamma}_k$. The feasible version is obtained by replacing these elements by nonparametric kernel estimates of the components of $\boldsymbol{\Omega}$ based on the OLS residuals in

[4] It can be shown that the correction term for Y_t is associated with the correction for the endogeneity bias while $\boldsymbol{\Delta}^{+}_{ku}$ eliminates the non-centrality bias.

(2.7), that are consistent under the assumption of cointegration, and requires the choice of the bandwidth to ensure the proper asymptotic correction for serial correlation and endogeneity.[5]

Remark 2.2. Using (2.1) we have that $\mathbf{Z}_{k,t} = \Delta\mathbf{X}_{k,t} = \mathbf{\Phi}_{k,p}\boldsymbol{\tau}_{p,t} + \boldsymbol{\varepsilon}_{k,t}$. By OLS detrending we have $\hat{\mathbf{Z}}_{kt,p} = \mathbf{Z}_{k,t} - \sum_{j=1}^{n}\mathbf{Z}_{k,j}\boldsymbol{\tau}'_{p,j}\mathbf{Q}^{-1}_{n,pp}\boldsymbol{\tau}_{p,t} = \boldsymbol{\varepsilon}_{k,t} - \sum_{j=1}^{n}\boldsymbol{\varepsilon}_{k,j}\boldsymbol{\tau}'_{p,j}\mathbf{Q}^{-1}_{n,pp}\boldsymbol{\tau}_{p,t} = \boldsymbol{\varepsilon}_{kt,p}$. If we define now Y_t^+ as $Y_t^+ = Y_t - \hat{\mathbf{Z}}'_{kt,p}\boldsymbol{\gamma}_k = Y_t - \boldsymbol{\varepsilon}'_{kt,p}\boldsymbol{\gamma}_k$, as indicated by Hansen (1992) (page 93), then the FM-OLS estimator of $\boldsymbol{\alpha}_p$ and $\boldsymbol{\beta}_k$ is given by

$$\begin{pmatrix}\hat{\boldsymbol{\alpha}}^+_{p,n}\\ \hat{\boldsymbol{\beta}}^+_{k,n}\end{pmatrix} = \begin{pmatrix}\hat{\boldsymbol{\alpha}}_{p,n}\\ \hat{\boldsymbol{\beta}}_{k,n}\end{pmatrix} - (\mathbf{W}'_n)^{-1}\begin{pmatrix}\mathbf{Q}_{n,pp} & \mathbf{Q}_{n,pk}\\ \mathbf{Q}'_{n,pk} & \mathbf{Q}_{n,kk}\end{pmatrix}^{-1}\left\{\sum_{t=1}^{n}\begin{pmatrix}\boldsymbol{\tau}_{p,tn}\\ \boldsymbol{\eta}_{k,tn}\end{pmatrix}\boldsymbol{\varepsilon}'_{kt,p}\cdot\boldsymbol{\gamma}_k + \mathbf{W}_n^{-1}\begin{pmatrix}\mathbf{0}_{p+1}\\ n\Delta^+_{ku}\end{pmatrix}\right\}$$

which gives exactly the same expressions as before for $(\hat{\boldsymbol{\alpha}}^+_{p,n}, \hat{\boldsymbol{\beta}}^+_{k,n})$ and the FM-OLS residuals.

For later use, we define the partial sum of the detrended errors in the cointegrating regression (2.3) or (2.8) as $U_{t,p} = \sum_{j=1}^{t} u_{j,p}$, with

$$n^{-(1-v)}U_{[nr],p} = n^{-(1-v)}\sum_{t=1}^{[nr]}u_{t,p} = n^{-(1-v)}\sum_{t=1}^{[nr]}u_t - n^{-1}\sum_{t=1}^{[nr]}\boldsymbol{\tau}'_{p,tn}\bar{\mathbf{Q}}^{-1}_{n,pp}n^{-(1-v)}\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}u_t$$

where, asymptotically we have

$$n^{-(1-v)}U_{[nr],p} \Rightarrow J_{u,p}(r) = \begin{cases} V_{u,p}(r) & v = 1/2 \quad (|\alpha| < 1)\\ \int_0^r B_{u,p}(s)ds & v = -1/2\end{cases} \tag{2.14}$$

with $V_{u,p}(r)$ a generalized (p+1)th-level Brownian bridge process given by

$$V_{u,p}(r) = B_u(r) - \int_0^r \boldsymbol{\tau}'_p(s)ds\,\mathbf{Q}^{-1}_{pp}\int_0^1 \boldsymbol{\tau}_p(s)dB_u(s) \tag{2.15}$$

[5] Similarly, since the CCR estimator proposed by Park (1992) is defined as the OLS estimator between the modified dependent variable $Y_t^* = Y_t - (\hat{\boldsymbol{\beta}}'_{k,n}\Delta_k\boldsymbol{\Sigma}^{-1} + (0,\boldsymbol{\gamma}'_k))\hat{\boldsymbol{\xi}}_{t,p}(k)$ and $(\boldsymbol{\tau}'_{p,t}, \mathbf{X}'^*_{k,t})'$, with $\mathbf{X}^*_{k,t} = \mathbf{X}_{k,t} - \Delta_k\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\xi}}_{t,p}(k)$, $\hat{\boldsymbol{\xi}}_{t,p}(k) = (u'_{t,p}(k), \mathbf{Z}'_{k,t})'$, and $\Delta_k = \sum_{j=0}^{\infty}E[\boldsymbol{\varepsilon}_{k,t}\boldsymbol{\xi}'_{t-j}] = \sum_{j=0}^{\infty}E[\boldsymbol{\varepsilon}_{k,t}(u_{t-j}, \boldsymbol{\varepsilon}'_{k,t-j})]$. This method uses the same principle as the FM-OLS method to eliminate the endogeneity bias, while it deals with the non-centrality parameter in a different manner, but also relies on consistent estimates of the quantities $\boldsymbol{\Delta}_k$, $\boldsymbol{\Sigma}$ and $\boldsymbol{\gamma}_k$ which depend on some tuning parameters.

with variance $E[V_{u,p}(r)^2] = \omega_u^2 \cdot b_p(r)$, where $b_p(r) = r - \int_0^r \boldsymbol{\tau}_p'(s)ds\,\mathbf{Q}_{pp}^{-1}\int_0^r \boldsymbol{\tau}_p(s)ds$, and $B_{u,p}(r)$ a (p+1)th-order detrended Brownian motion process defined as

$$B_{u,p}(r) = B_u(r) - \boldsymbol{\tau}_p'(r)\mathbf{Q}_{pp}^{-1}\int_0^1 \boldsymbol{\tau}_p(s)B_u(s)ds \tag{2.16}$$

as the stochastic limits in (2.14).[6] Thus, under the assumption of no cointegration, when $\alpha = 1$ and $v = -1/2$, we get the usual result

$$\hat{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k \Rightarrow \left(\int_0^1 \mathbf{B}_{k,p}(s)\mathbf{B}_{k,p}(s)'ds\right)^{-1}\int_0^1 \mathbf{B}_{k,p}(s)dJ_{u,p}(s)$$

where $dJ_{u,p}(r) = B_{u,p}(r)$. Finally, making use of (2.15), and the relation $B_u(r) = B_{u.k}(r) + \boldsymbol{\gamma}_k'\mathbf{B}_k(r)$ we then have that $V_{u,p}(r)$ can be decomposed as

$$\begin{aligned} V_{u,p}(r) &= B_{u.k}(r) - \int_0^r \boldsymbol{\tau}_p'(s)ds\,\mathbf{Q}_{pp}^{-1}\int_0^1 \boldsymbol{\tau}_p(s)dB_{u.k}(s) \\ &\quad + \boldsymbol{\gamma}_k'\left\{\mathbf{B}_k(r) - \int_0^1 d\mathbf{B}_k(s)\boldsymbol{\tau}_p(s)'\mathbf{Q}_{pp}^{-1}\int_0^r \boldsymbol{\tau}_p'(s)ds\right\} \\ &= V_{u.k,p}(r) + \boldsymbol{\gamma}_k'\mathbf{V}_{k,p}(r) \end{aligned} \tag{2.17}$$

where, by construction, it is verified $E[\mathbf{V}_{k,p}(r)V_{u.k,p}(r)] = E[\mathbf{B}_{k,p}(r)V_{u.k,p}(r)] = \mathbf{0}_k$, with $\mathbf{B}_{k,p}(r)$ defined in (2.9), and $E[V_{u.k,p}(r)^2] = \omega_{u.k}^2 \cdot b_p(r)$.

3. IM-OLS estimation with trending regressors

In this section we consider the new estimator of a static cointegrating regression model like (2.3) recently proposed by Vogelsang and Wagner (2011). For this estimator, these authors show that a simple transformation of the model components is used to obtain an asymptotically unbiased estimator of $\boldsymbol{\beta}_k$ with a zero mean Gaussian mixture limiting distribution, but when the assumed DGP is as in (2.1) with $\mathbf{A}_{k,p} = \mathbf{0}_{k,p+1}$. Like FM-OLS,

[6] Explicit expressions for these two limiting processes, $V_{u,p}(r)$ and $B_{u,p}(r)$, can be obtained in the leading cases of p = 0 (constant), and p = 1 (constant and linear trend). Specifically, we have that $V_{u,0}(r) = B_u(r) - rB_u(1)$, and $V_{u,1}(r) = B_u(r) + (2-3r)rB_u(1) - 6r(1-r)\int_0^1 B_u(s)ds$ for the first and second-level Brownian bridge, while that $B_{u,0}(r) = B_u(r) - \int_0^1 B_u(s)ds$, and $B_{u,1}(r) = B_u(r) + 2(3r-2)\int_0^1 B_u(s)ds$ $-2(6r-3)\int_0^1 sB_u(s)ds$ are the particular expressions for the demeaned and demeaned and detrended Brownian processes, respectively.

the transformation has two steps but neither step requires the estimation of any of the components of $\mathbf{\Omega}$, and so the choice of bandwidth and kernel is completely avoided. Thus, computing the partial sum of both sides of (2.3) gives the so-called integrated cointegrating regression model as

$$S_t = \boldsymbol{\alpha}'_p \mathbf{S}_{p,t} + \boldsymbol{\beta}'_k \mathbf{S}_{k,t} + U_t \quad t = 1,...,n \tag{3.1}$$

with $U_t = \sum_{j=1}^t u_j$, $S_t = \sum_{j=1}^t Y_j = \boldsymbol{\alpha}'_{0,p} \sum_{j=1}^t \boldsymbol{\tau}_{p,j} + \sum_{j=1}^t \eta_{0,j} = \boldsymbol{\alpha}'_{0,p} \mathbf{S}_{p,t} + h_{0,t}$,

$$\mathbf{S}_{p,t} = \sum_{j=1}^t \boldsymbol{\tau}_{p,j} = \boldsymbol{\Gamma}^{-1}_{p,n} \sum_{j=1}^t \boldsymbol{\tau}_{p,jn} = \boldsymbol{\Gamma}^{-1}_{p,n} \mathbf{S}_{p,tn} \tag{3.2}$$

and

$$\mathbf{S}_{k,t} = \sum_{j=1}^t \mathbf{X}_{k,j} = \mathbf{A}_{k,p} \mathbf{S}_{p,t} + \sum_{j=1}^t \boldsymbol{\eta}_{k,j} = \mathbf{A}_{k,p} \boldsymbol{\Gamma}^{-1}_{p,n} \mathbf{S}_{p,tn} + \mathbf{H}_{k,t} \tag{3.3}$$

Taking together (3.2) and (3.3) we have

$$\begin{pmatrix} \mathbf{S}_{p,t} \\ \mathbf{S}_{k,t} \end{pmatrix} = \begin{pmatrix} n\boldsymbol{\Gamma}^{-1}_{p,n} & \mathbf{0}_{p+1,k} \\ n\mathbf{A}_{k,p}\boldsymbol{\Gamma}^{-1}_{p,n} & n\sqrt{n}\mathbf{I}_{k,k} \end{pmatrix} \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \end{pmatrix} = \mathbf{W}_{1,n} \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \end{pmatrix}$$

with

$$\begin{pmatrix} n^{-1}\mathbf{S}_{p,[nr]n} \\ n^{-3/2}\mathbf{H}_{k,[nr]} \end{pmatrix} \Rightarrow \mathbf{g}(r) = \begin{pmatrix} \int_0^r \boldsymbol{\tau}_p(s)ds \\ \int_0^r \mathbf{B}_k(s)ds \end{pmatrix} \tag{3.4}$$

as $n\rightarrow\infty$. Then, the IM-OLS estimator of $\boldsymbol{\alpha}_p$ and $\boldsymbol{\beta}_k$ is given by the OLS estimator in (3.1), which can be written as

$$\begin{aligned} n^{-(1-v)}\mathbf{W}'_{1,n} \begin{pmatrix} \tilde{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p \\ \tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k \end{pmatrix} &= \begin{pmatrix} n^v \boldsymbol{\Gamma}^{-1}_{p,n}[(\tilde{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p) + \mathbf{A}'_{k,p}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k)] \\ n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \end{pmatrix} \\ &= \left((1/n)\sum_{t=1}^n \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \end{pmatrix} (n^{-1}\mathbf{S}'_{p,tn}, n^{-3/2}\mathbf{H}'_{k,t}) \right)^{-1} \\ &\quad \times \left\{ (1/n)\sum_{t=1}^n \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \end{pmatrix} n^{-(1-v)} U_t \right\} \end{aligned} \tag{3.5}$$

where

$$(1/n)\sum_{t=1}^n \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \end{pmatrix} n^{-(1-v)} U_t \Rightarrow \int_0^1 \mathbf{g}(r) J_u(r) dr \tag{3.6}$$

with $\mathbf{g}(r)$ given in (3.4), $J_u(r) = B_u(r)$ under the assumption of cointegration, v = 1/2, and $J_u(r) = \int_0^r B_u(s)ds$ under no cointegration, that is when v = −1/2. Vogelsang and Wagner (2011), Theorem 2, considers this case when $\mathbf{A}_{k,p} = \mathbf{0}_{k,p+1}$, which corresponds to the case of integrated regressors without deterministically trending components, so that the trending parameters in the specification of the cointegrating regression must be associated to the deterministic component of the dependent variable. Given that the limiting result in (3.6) does not contain the additive term $\mathbf{\Delta}_{ku}$, partial summing before estimating the model thus performs the same role for IM-OLS that $(\mathbf{0}'_{p+1}, n\mathbf{\Delta}'^{+}_{ku})'$ plays for FM-OLS, but this still leaves the problem that the correlation between u_t and $\boldsymbol{\varepsilon}_{k,t}$ rules out the possibility of conditioning on $\mathbf{B}_k(r)$ to obtain a conditional asymptotic normality result. The solution proposed by these authors requires that $\mathbf{X}_{k,t}$ be added as a regressor to the partial sum regression (3.1) as

$$S_t = \boldsymbol{\alpha}'_p \mathbf{S}_{p,t} + \boldsymbol{\beta}'_k \mathbf{S}_{k,t} + \boldsymbol{\gamma}'_k \mathbf{X}_{k,t} + \zeta_t \quad t = 1, ..., n \tag{3.7}$$

which can be called now the integrated modified (IM) cointegrating regression, where $\zeta_t = U_t - \boldsymbol{\gamma}'_k \mathbf{X}_{k,t}$. Then, by OLS estimation of (3.7) we have that

$$\begin{pmatrix} n^v \mathbf{\Gamma}^{-1}_{p,n}(\tilde{\boldsymbol{\alpha}}_{p,n} - \boldsymbol{\alpha}_p) \\ n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ n^{-1/2+v}\tilde{\boldsymbol{\gamma}}_{k,n} \end{pmatrix} = \left((1/n)\sum_{t=1}^{n} \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{S}_{k,t} \\ n^{-1/2}\mathbf{X}_{k,t} \end{pmatrix} (n^{-1}\mathbf{S}'_{p,tn}, n^{-3/2}\mathbf{S}'_{k,t}, n^{-1/2}\mathbf{X}'_{k,t}) \right)^{-1} \times (1/n)\sum_{t=1}^{n} \left\{ \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{S}_{k,t} \\ n^{-1/2}\mathbf{X}_{k,t} \end{pmatrix} n^{-(1-v)} U_t \right\} \tag{3.8}$$

which gives a well defined limit result and free of nuisance trending parameters under the assumption of cointegration and no deterministic component in the DGP (2.1) for $\mathbf{X}_{k,t}$, as in Theorem 2 in Vogelsang and Wagner (2011). However, when using $\mathbf{X}_{k,t} = \mathbf{A}_{k,p}\mathbf{\Gamma}^{-1}_{p,n}\boldsymbol{\tau}_{p,tn} + \sqrt{n}(n^{-1/2}\boldsymbol{\eta}_{k,t})$ from (2.1) and (2.4), with $\boldsymbol{\tau}_{p,tn} = \mathbf{S}_{p,tn} - \mathbf{S}_{p,(t-1)n}$, then we can write

$$\begin{pmatrix} \mathbf{S}_{p,t} \\ \mathbf{S}_{k,t} \\ \mathbf{X}_{k,t} \end{pmatrix} = \begin{pmatrix} n\boldsymbol{\Gamma}_{p,n}^{-1}(n^{-1}\mathbf{S}_{p,tn}) \\ n\mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1}(n^{-1}\mathbf{S}_{p,tn}) + n\sqrt{n}(n^{-3/2}\mathbf{H}_{k,t}) \\ n\mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1}(n^{-1}\mathbf{S}_{p,tn}) + \sqrt{n}(n^{-1/2}\boldsymbol{\eta}_{k,t}) \end{pmatrix} - \begin{pmatrix} \mathbf{0}_{p+1} \\ \mathbf{0}_k \\ n\mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1}(n^{-1}\mathbf{S}_{p,(t-1)n}) \end{pmatrix}$$

$$= \begin{pmatrix} n\boldsymbol{\Gamma}_{p,n}^{-1} & \mathbf{0}_{p+1,k} & \mathbf{0}_{p+1,k} \\ n\mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1} & n\sqrt{n}\mathbf{I}_{k,k} & \mathbf{0}_{k,k} \\ n\mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1} & \mathbf{0}_{k,k} & \sqrt{n}\mathbf{I}_{k,k} \end{pmatrix} \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix}$$

$$- \begin{pmatrix} \mathbf{0}_{p+1,p+1} & \mathbf{0}_{p+1,k} & \mathbf{0}_{p+1,k} \\ \mathbf{0}_{k,p+1} & \mathbf{0}_{k,k} & \mathbf{0}_{k,k} \\ n\mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1} & \mathbf{0}_{k,k} & \mathbf{0}_{k,k} \end{pmatrix} \begin{pmatrix} n^{-1}\mathbf{S}_{p,(t-1)n} \\ n^{-3/2}\mathbf{H}_{k,t-1} \\ n^{-1/2}\boldsymbol{\eta}_{k,t-1} \end{pmatrix}$$

that is,

$$\begin{pmatrix} \mathbf{S}_{p,t} \\ \mathbf{S}_{k,t} \\ \mathbf{X}_{k,t} \end{pmatrix} = \mathbf{W}_{21,n} \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} - \mathbf{W}_{22,n} \begin{pmatrix} n^{-1}\mathbf{S}_{p,(t-1)n} \\ n^{-3/2}\mathbf{H}_{k,t-1} \\ n^{-1/2}\boldsymbol{\eta}_{k,t-1} \end{pmatrix}$$

$$= \mathbf{W}_{21,n} \left\{ \begin{pmatrix} n^{-1}\mathbf{S}_{p,tn} \\ n^{-3/2}\mathbf{H}_{k,t} \\ n^{-1/2}\boldsymbol{\eta}_{k,t} \end{pmatrix} - \mathbf{W}_{21,n}^{-1}\mathbf{W}_{22,n} \begin{pmatrix} n^{-1}\mathbf{S}_{p,(t-1)n} \\ n^{-3/2}\mathbf{H}_{k,t-1} \\ n^{-1/2}\boldsymbol{\eta}_{k,t-1} \end{pmatrix} \right\}$$

where the last term between brackets is given by

$$\mathbf{W}_{21,n}^{-1}\mathbf{W}_{22,n} \begin{pmatrix} n^{-1}\mathbf{S}_{p,(t-1)n} \\ n^{-3/2}\mathbf{H}_{k,t-1} \\ n^{-1/2}\boldsymbol{\eta}_{k,t-1} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{p+1} \\ \mathbf{0}_k \\ \sqrt{n}\mathbf{A}_{k,p}\boldsymbol{\Gamma}_{p,n}^{-1}(n^{-1}\mathbf{S}_{p,(t-1)n}) \end{pmatrix}$$

which diverge with the sample size even in the case of a constant term (p = 0). Alternatively, if we redefine the IM regression model (3.7) in terms of the IM-OLS detrended variables we have

$$\hat{S}_{t,p}^* = \boldsymbol{\beta}_k'\hat{\mathbf{S}}_{kt,p}^* + \boldsymbol{\gamma}_k'\hat{\mathbf{X}}_{kt,p}^* + \zeta_{t,p}^* \quad t = 1,\ldots,n$$

where

$$\begin{pmatrix} \hat{S}_{t,p}^* \\ \hat{\mathbf{S}}_{kt,p}^* \\ \hat{\mathbf{X}}_{kt,p}^* \end{pmatrix} = \begin{pmatrix} S_t \\ \mathbf{S}_{k,t} \\ \mathbf{X}_{k,t} \end{pmatrix} - \sum_{j=1}^{n} \begin{pmatrix} S_j \\ \mathbf{S}_{k,j} \\ \mathbf{X}_{k,j} \end{pmatrix} \mathbf{S}_{p,j}' \left(\sum_{j=1}^{n} \mathbf{S}_{p,j}\mathbf{S}_{p,j}' \right)^{-1} \mathbf{S}_{p,t}$$

Given $S_t = \boldsymbol{\alpha}_{0,p}'\mathbf{S}_{p,t} + h_{0,t}$ and $\mathbf{S}_{k,t}$ as in (3.3) we then have that

$$\begin{pmatrix} \hat{S}^*_{t,p} \\ \hat{\mathbf{S}}^*_{kt,p} \end{pmatrix} = \begin{pmatrix} h_{0,t} \\ \mathbf{H}_{k,t} \end{pmatrix} - \sum_{j=1}^{n} \begin{pmatrix} h_{0,j} \\ \mathbf{H}_{k,j} \end{pmatrix} \mathbf{S}'_{p,j} \left(\sum_{j=1}^{n} \mathbf{S}_{p,j} \mathbf{S}'_{p,j} \right)^{-1} \mathbf{S}_{p,t} = \begin{pmatrix} h^*_{0t,p} \\ \mathbf{H}^*_{kt,p} \end{pmatrix}$$

which are free of trend parameters, while that $\hat{\mathbf{X}}^*_{kt,p}$ is given by

$$\begin{aligned} \hat{\mathbf{X}}^*_{kt,p} &= \boldsymbol{\eta}_{k,t} - \sum_{j=1}^{n} \boldsymbol{\eta}_{k,j} \mathbf{S}'_{p,j} \left(\sum_{j=1}^{n} \mathbf{S}_{p,j} \mathbf{S}'_{p,j} \right)^{-1} \mathbf{S}_{p,t} \\ &\quad + \mathbf{A}_{k,p} \left(\boldsymbol{\tau}_{p,t} - \sum_{j=1}^{n} \boldsymbol{\tau}_{p,j} \mathbf{S}'_{p,j} \left(\sum_{j=1}^{n} \mathbf{S}_{p,j} \mathbf{S}'_{p,j} \right)^{-1} \mathbf{S}_{p,t} \right) = \boldsymbol{\eta}^*_{kt,p} + \mathbf{A}_{k,p} \boldsymbol{\tau}^*_{p,t} \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{\eta}^*_{kt,p} &= \sqrt{n} \left\{ (n^{-1/2} \boldsymbol{\eta}_{k,t}) - (1/n) \sum_{j=1}^{n} (n^{-1/2} \boldsymbol{\eta}_{k,j})(n^{-1} \mathbf{S}'_{p,j}) \right. \\ &\quad \left. \times \left((1/n) \sum_{j=1}^{n} (n^{-1} \mathbf{S}_{p,jn})(n^{-1} \mathbf{S}'_{p,jn}) \right)^{-1} (n^{-1} \mathbf{S}_{p,tn}) \right\} \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}_{k,p} \boldsymbol{\tau}^*_{p,t} &= \mathbf{A}_{k,p} \boldsymbol{\Gamma}^{-1}_{p,n} \left\{ \boldsymbol{\tau}_{p,tn} - (1/n) \sum_{j=1}^{n} \boldsymbol{\tau}_{p,jn} (n^{-1} \mathbf{S}'_{p,jn}) \right. \\ &\quad \left. \times \left((1/n) \sum_{j=1}^{n} (n^{-1} \mathbf{S}_{p,jn})(n^{-1} \mathbf{S}'_{p,jn}) \right)^{-1} (n^{-1} \mathbf{S}_{p,tn}) \right\} = \mathbf{A}_{k,p} \boldsymbol{\Gamma}^{-1}_{p,n} \boldsymbol{\tau}^*_{p,tn} \end{aligned}$$

which determines that $n^{-1/2} \hat{\mathbf{X}}^*_{kt,p} = n^{-1/2} \boldsymbol{\eta}^*_{kt,p} + \mathbf{A}_{k,p} (n^{-1/2} \boldsymbol{\Gamma}^{-1}_{p,n}) \boldsymbol{\tau}^*_{p,tn}$. For $p = 0$, $n^{-1/2} \hat{\mathbf{X}}^*_{kt,0} = n^{-1/2} \boldsymbol{\eta}^*_{kt,0} + \mathbf{A}_{k,0} n^{-1/2} \boldsymbol{\tau}^*_{0,tn} = n^{-1/2} \boldsymbol{\eta}^*_{kt,0} + O(n^{-1/2})$, so that the deterministic component is asymptotically irrelevant, while for $p \geq 1$ we have that $n^{-1/2} \hat{\mathbf{X}}^*_{kt,p} = n^{-1/2} \boldsymbol{\eta}^*_{kt,p} + O(n^{p-1/2})$, which implies that deterministic component dominates the stochastic one yielding inconsistent results. Thus, to deal with this general case, from (2.8) and making use of the result in Remark 2.2 for the OLS detrended observations of $\Delta \mathbf{X}_{k,t}$, $\hat{\mathbf{Z}}_{kt,p} = \boldsymbol{\varepsilon}_{kt,p}$, we get the following augmented version of (2.8)

$$\hat{Y}_{t,p} = \boldsymbol{\beta}'_k \hat{\mathbf{X}}_{kt,p} + \boldsymbol{\gamma}'_k \hat{\mathbf{Z}}_{kt,p} + u_{t,p} - \boldsymbol{\gamma}'_k \hat{\mathbf{Z}}_{kt,p} = \boldsymbol{\beta}'_k \hat{\mathbf{X}}_{kt,p} + \boldsymbol{\gamma}'_k \hat{\mathbf{Z}}_{kt,p} + z_{t,p}, \quad t = 1, \ldots, n$$

which gives the following corrected version of the IM cointegrating regression equation

$$\hat{S}_{t,p} = \boldsymbol{\beta}_k'\hat{\mathbf{S}}_{kt,p} + \boldsymbol{\gamma}_k'\sum_{j=1}^{t}\hat{\mathbf{Z}}_{kj,p} + \zeta_{t,p} = \boldsymbol{\beta}_k'\hat{\mathbf{S}}_{kt,p} + \boldsymbol{\gamma}_k'\hat{\mathbf{T}}_{kt,p} + \zeta_{t,p}, \quad t = 1,...,n \tag{3.9}$$

with $\hat{\mathbf{S}}_{kt,p} = \sum_{j=1}^{t}\hat{\mathbf{X}}_{kj,p} = \sum_{j=1}^{t}\boldsymbol{\eta}_{kj,p}$, and $\zeta_{t,p} = \sum_{j=1}^{t} z_{t,p} = U_{t,p} - \boldsymbol{\gamma}_k'\hat{\mathbf{T}}_{kt,p}$, where

$$\begin{pmatrix}\hat{\mathbf{S}}_{kt,p}\\ \hat{\mathbf{T}}_{kt,p}\end{pmatrix} = \begin{pmatrix} n\sqrt{n}\mathbf{I}_{k,k} & \mathbf{0}_{k,k}\\ \mathbf{0}_{k,k} & \sqrt{n}\mathbf{I}_{k,k}\end{pmatrix}\begin{pmatrix} n^{-3/2}\hat{\mathbf{S}}_{kt,p}\\ n^{-1/2}\hat{\mathbf{T}}_{kt,p}\end{pmatrix} = \mathbf{W}_n\begin{pmatrix} n^{-3/2}\hat{\mathbf{S}}_{kt,p}\\ n^{-1/2}\hat{\mathbf{T}}_{kt,p}\end{pmatrix} \tag{3.10}$$

and

$$\begin{aligned}\hat{\mathbf{T}}_{k[nr],p} &= \sum_{t=1}^{[nr]}\hat{\mathbf{Z}}_{kt,p} = \sum_{t=1}^{[nr]}\boldsymbol{\varepsilon}_{k,t} - \sum_{j=1}^{n}\boldsymbol{\varepsilon}_{k,j}\boldsymbol{\tau}_{p,jn}'\,\mathbf{Q}_{n,pp}^{-1}\sum_{t=1}^{[nr]}\boldsymbol{\tau}_{p,tn}\\ &= -\boldsymbol{\eta}_{k,0} + \sqrt{n}\left(n^{-1/2}\boldsymbol{\eta}_{k,[nr]} - n^{-1/2}\sum_{j=1}^{n}\boldsymbol{\varepsilon}_{k,j}\boldsymbol{\tau}_{p,jn}'\,\bar{\mathbf{Q}}_{n,pp}^{-1}(1/n)\sum_{t=1}^{[nr]}\boldsymbol{\tau}_{p,tn}\right)\end{aligned}$$

which gives, asymptotically, a k-dimensional Brownian bridge of order (p+1) such that

$$n^{-1/2}\hat{\mathbf{T}}_{k[nr],p} \Rightarrow \mathbf{V}_{k,p}(r) = \mathbf{B}_k(r) - \int_0^1 d\mathbf{B}_k(s)\boldsymbol{\tau}_p'(s)\,\mathbf{Q}_{pp}^{-1}\int_0^r \boldsymbol{\tau}_p(s)ds$$

or, more compactly,

$$\begin{pmatrix} n^{-3/2}\hat{\mathbf{S}}_{k[nr],p}\\ n^{-1/2}\hat{\mathbf{T}}_{k[nr],p}\end{pmatrix} \Rightarrow \mathbf{g}_p(r) = \begin{pmatrix}\int_0^r \mathbf{B}_{k,p}(s)ds\\ \mathbf{V}_{k,p}(r)\end{pmatrix} \tag{3.11}$$

Also, from (2.14), (2.17), and (3.11) it can be easily verified that under cointegration ($|\alpha| < 1$ in Assumption 2.1), the scaled error term in the IM cointegrating regression (3.9) behaves asymptotically as

$$n^{-1/2}\zeta_{t,p} = n^{-1/2}U_{t,p} - \boldsymbol{\gamma}_k' n^{-1/2}\hat{\mathbf{T}}_{kt,p} \Rightarrow V_{u.k,p}(r). \tag{3.12}$$

Then, we define the IM-OLS estimator of the coefficient vector ($\boldsymbol{\beta}_k$, $\boldsymbol{\gamma}_k$), based on OLS detrended observations, as

$$\begin{pmatrix}\tilde{\boldsymbol{\beta}}_{k,n}\\ \tilde{\boldsymbol{\gamma}}_{k,n}\end{pmatrix} = \left(\sum_{t=1}^{n}\begin{pmatrix}\hat{\mathbf{S}}_{kt,p}\\ \hat{\mathbf{T}}_{kt,p}\end{pmatrix}(\hat{\mathbf{S}}_{kt,p}', \hat{\mathbf{T}}_{kt,p}')\right)^{-1}\sum_{t=1}^{n}\begin{pmatrix}\hat{\mathbf{S}}_{kt,p}\\ \hat{\mathbf{T}}_{kt,p}\end{pmatrix}\hat{S}_{t,p} \tag{3.13}$$

with IM-OLS residual sequence given by

$$\tilde{\zeta}_{t,p}(k) = \hat{S}_{t,p} - (\hat{\mathbf{S}}_{kt,p}', \hat{\mathbf{T}}_{kt,p}')\begin{pmatrix}\tilde{\boldsymbol{\beta}}_{k,n}\\ \tilde{\boldsymbol{\gamma}}_{k,n}\end{pmatrix} \qquad t = 1, ..., n \tag{3.14}$$

Next proposition establish the main result in this section related to the weak convergence of IM-OLS estimators and residuals under the assumption of cointegration, that is when the error term sequence u_t in the original cointegrating regression equation (2.3) is nonstationary with $\alpha = 1$ in Assumption 2.1.

Proposition 3.1. *Given (2.1) and (2.2), and under Assumption 2.1, the IM-OLS estimation of the cointegrating regression model in (2.3) based on the IM regression (3.9) with OLS detrended observations, then equation (3.13) determine that:*

$$(a)\begin{pmatrix} n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k) \\ n^{-1/2+v}\tilde{\boldsymbol{\gamma}}_{k,n} \end{pmatrix} = \left((1/n)\sum_{t=1}^{n} \begin{pmatrix} n^{-3/2}\hat{\mathbf{S}}_{kt,p} \\ n^{-1/2}\hat{\mathbf{T}}_{kt,p} \end{pmatrix} (n^{-3/2}\hat{\mathbf{S}}'_{kt,p}, n^{-1/2}\hat{\mathbf{T}}'_{kt,p}) \right)^{-1} \times (1/n)\sum_{t=1}^{n} \left\{ \begin{pmatrix} n^{-3/2}\hat{\mathbf{S}}_{kt,p} \\ n^{-1/2}\hat{\mathbf{T}}_{kt,p} \end{pmatrix} n^{-(1-v)}U_{t,p} \right\}$$

$$(b)\begin{pmatrix} n(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k) \\ \tilde{\boldsymbol{\gamma}}_{k,n}-\boldsymbol{\gamma}_k \end{pmatrix} \Rightarrow \left(\int_0^1 \mathbf{g}_p(r)\mathbf{g}_p(r)'dr\right)^{-1} \int_0^1 \mathbf{g}_p(r)V_{u.k,p}(r)dr$$
$$= \omega_{u.k}\boldsymbol{\Pi}^{-1}\left(\int_0^1 \bar{\mathbf{g}}_p(r)\bar{\mathbf{g}}_p(r)'dr\right)^{-1} \int_0^1 \bar{\mathbf{g}}_p(r)W_{u.k,p}(r)dr$$
$$= \omega_{u.k}\boldsymbol{\Pi}^{-1}\left(\int_0^1 \bar{\mathbf{g}}_p(r)\bar{\mathbf{g}}_p(r)'dr\right)^{-1} \int_0^1 [\bar{\mathbf{G}}_p(1)-\bar{\mathbf{G}}_p(r)]dW_{u.k,p}(r)$$

$$(c)\, n^{-(1-v)}\tilde{\zeta}_{t,p}(k) = n^{-(1-v)}\zeta_{t,p} - (n^{-3/2}\hat{\mathbf{S}}'_{kt,p}, n^{-1/2}\hat{\mathbf{T}}'_{kt,p})\begin{pmatrix} n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k) \\ n^{-1/2+v}(\tilde{\boldsymbol{\gamma}}_{k,n}-\boldsymbol{\gamma}_k) \end{pmatrix}$$

and

$$(d)\, n^{-1/2}\tilde{\zeta}_{t,p}(k) \Rightarrow \omega_{u.k}\left\{ W_{u.k,p}(r) - \bar{\mathbf{g}}_p(r)'\left(\int_0^1 \bar{\mathbf{g}}_p(s)\bar{\mathbf{g}}_p(s)'ds\right)^{-1} \times \int_0^1 [\bar{\mathbf{G}}_p(1)-\bar{\mathbf{G}}_p(s)]dW_{u.k,p}(s) \right\} = \omega_{u.k}R_{k,p}(r)$$

where the results in (b) and (d) are establish under the assumption of cointegration, that is with v = 1/2, with $V_{u.k,p}(r) = \omega_{u.k}W_{u.k,p}(r)$, $\mathbf{g}_p(r) = \boldsymbol{\Pi}\cdot\bar{\mathbf{g}}_p(r)$, $\mathbf{G}_p(r) = \int_0^r \mathbf{g}_p(s)ds = \boldsymbol{\Pi}\bar{\mathbf{G}}_p(r)$, *and* $\boldsymbol{\Pi} = diag(\boldsymbol{\Omega}_{k,k}^{1/2}, \boldsymbol{\Omega}_{k,k}^{1/2})$.

Proof. See Appendix C.

Remark 3.1. As can be seen in (b) and (d) above, for inferential purposes related to hypothesis testing, these limiting results depends only on $\omega_{u.k}^2$ and $\boldsymbol{\Omega}_{k,k}$ as nuisance parameters. Specially relevant, when using the IM-OLS residuals in (d), is the question of possible consistent estimation of the conditional long-run variance $\omega_{u.k}^2$ based on the first differences of $\tilde{\zeta}_{t,p}(k)$, $\Delta\tilde{\zeta}_{t,p}(k)$. As is discussed in the next section and in

Vogelsang and Wagner (2011), the standard approach based on the use of a nonparametric kernel-type estimator determine inconsistent estimation of $\omega_{u.k}^2$. For this, reason, for our purposes in this paper we follow an alternative approach.

4. IM-OLS residual-based test for the null of cointegration

In this section we propose some new statistics based on the sequence of IM-OLS residuals, as has been defined in section 3, for testing the null hypothesis of cointegration against the alternative of no cointegration by looking for excessive fluctuations in the sample paths of this residual sequence . These new test statistics are partially inspired by the nonparametric variance-ratio statistic proposed by Breitung (2002) for testing the unit root null hypothesis against stationarity in a univariate time series, in the sense that our statistics are totally free of tuning parameters. In our case, we look for a unit root-like behavior in the residual sequence $\tilde{\zeta}_{t,p}(k)$ which is compatible with the stationarity of the error term $z_{t,p}$ in the augmented cointegrating regression among the OLS detrended variables.

First of all, we consider the case of the IM cointegrating regressión (3.7) with $\boldsymbol{\alpha}_p = \mathbf{0}_{p+1}$ and $\mathbf{A}_{k,p} = \mathbf{0}_{k,p+1}$. Thus, from (3.8) we have that the IM-OLS estimators of $\boldsymbol{\beta}_k$ and $\boldsymbol{\gamma}_k$ can be written as

$$\begin{pmatrix} n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ n^{-1/2+v}\tilde{\boldsymbol{\gamma}}_{k,n} \end{pmatrix} = \left((1/n)\sum_{t=1}^{n} \begin{pmatrix} n^{-3/2}\mathbf{S}_{k,t} \\ n^{-1/2}\mathbf{X}_{k,t} \end{pmatrix} (n^{-3/2}\mathbf{S}'_{k,t}, n^{-1/2}\mathbf{X}'_{k,t}) \right)^{-1}$$
$$\times (1/n)\sum_{t=1}^{n} \left\{ \begin{pmatrix} n^{-3/2}\mathbf{S}_{k,t} \\ n^{-1/2}\mathbf{X}_{k,t} \end{pmatrix} n^{-(1-v)} U_t \right\}$$

where $(n^{-3/2}\mathbf{S}'_{k,t}, n^{-1/2}\mathbf{X}'_{k,t})' = (n^{-3/2}\mathbf{H}'_{k,t}, n^{-1/2}\boldsymbol{\eta}'_{k,t})' \Rightarrow \mathbf{g}(r)$ for $t = [nr]$, with $\mathbf{g}(r) = (\int_0^r \mathbf{B}_k(s)'ds, \mathbf{B}_k(r)')'$, as $n \to \infty$. Under the assumption of cointegration, the limiting distribution of these estimates is as in Proposition 3.1(b), with $\mathbf{g}_p(r)$ and $W_{u.k,p}(r)$ replaced by $\mathbf{g}(r)$ and $W_{u.k}(r)$, respectively. With the associated sequence of IM-OLS residuals, $\tilde{\zeta}_t(k) = S_t - (\mathbf{S}'_{k,t}\tilde{\boldsymbol{\beta}}_{k,n} + \mathbf{X}'_{k,t}\tilde{\boldsymbol{\gamma}}_{k,n})$, $t = 1, \ldots, n$, we define the following main components of our fluctuation test statistics

$$F_{1,n}(k) = (1/n)\sum_{t=1}^{n}((1/\sqrt{n})\tilde{\zeta}_t(k))^2 \tag{4.1}$$

$$F_{2,n}(k) = \max_{t=1,\ldots,n}(1/\sqrt{n})\,|\,\tilde{\zeta}_t(k)\,| \tag{4.2}$$

and

$$F_{3,n}(k) = \max_{t=1,\ldots,n}(1/\sqrt{n})\,|\,\tilde{\zeta}_t(k) - (t/n)\tilde{\zeta}_n(k)\,| \tag{4.3}$$

Taking into account the result (d) in Proposition 3.1, we have that, asymptotically, $n^{-1/2}\tilde{\zeta}_t(k) \Rightarrow \omega_{u.k} R_k(r)$ under the cointegration assumption, with

$$R_k(r) = W_{u.k}(r) - \bar{\mathbf{g}}(r)'\left(\int_0^1 \bar{\mathbf{g}}(s)\bar{\mathbf{g}}(s)'ds\right)^{-1}\int_0^1[\bar{\mathbf{G}}(1) - \bar{\mathbf{G}}(s)]dW_{u.k}(s) \tag{4.4}$$

In order to eliminate the nuisance parameter $\omega_{u.k}^2$ from the limiting null distributions of these statistics, we define the random element

$$\tilde{v}_n^2(k) = (1/n)\tilde{\zeta}_n^2(k) \tag{4.5}$$

which gives the normalized version of the above fluctuation statistics

$$\bar{F}_{1,n}(k) = \tilde{v}_n^{-2}(k)\cdot F_{1,n}(k) \tag{4.6}$$

and

$$\bar{F}_{j,n}(k) = \tilde{v}_n^{-1}(k)\cdot F_{j,n}(k)\,,\, j = 2,\, 3 \tag{4.7}$$

Taking into account the fluctuation statistics $F_{j,n}(k)$ in (4.1)-(4.3), as well as the normalized squared error $\tilde{P}_n^2(k)$, can also be written as

$$F_{1,n}(k) = n^{1-2v}\left\{(1/n)\sum_{t=1}^{n}(n^{-(1-v)}\tilde{\zeta}_t(k))^2\right\}$$

$$F_{2,n}(k) = n^{(1-2v)/2}\max_{t=1,\ldots,n}|\,n^{-(1-v)}\tilde{\zeta}_t(k)\,|$$

$$\tilde{v}_{n,k}^2 = (1/n)[n^{1-v}(n^{-(1-v)}\tilde{\zeta}_n(k))]^2 = n^{1-2v}[n^{-(1-v)}\tilde{\zeta}_n(k)]^2$$

and similarly for $F_{3,n}(k)$, then both the numerator and the denominator of the normalized test statistics in (4.6) and (4.7) are of the same order of magnitude under the null hypothesis of cointegration, as well as under the alternative of no cointegration (when $v = -1/2$), but with very different limiting distributions in each case. Similarly, in the case of the IM-OLS estimation of the cointegrating regression model based on OLS detrended observations of the variables, as was introduced in section 3, then we define the corresponding normalized fluctuation test statistics as

$$\overline{F}_{1,n}(p,k) = \tilde{v}_{n,p}^{-2}(k) \cdot F_{1,n}(p,k) \tag{4.8}$$

and

$$\overline{F}_{j,n}(p,k) = \tilde{v}_{n,p}^{-1}(k) \cdot F_{j,n}(p,k),\; j = 2, 3 \tag{4.9}$$

where

$$\tilde{v}_{n,p}^{2}(k) = (1/n)\tilde{\zeta}_{n,p}^{2}(k) \tag{4.10}$$

Next proposition establish the asymptotic null and alternative distribution of all these test statistics.

Proposition 4.1. *Under the null hypothesis of cointegration, that is when* $\alpha = 1$ *in Assumption 2.1 with* $v = 1/2$*, then:*

$$(a)\, F_{1,n}(k) \Rightarrow \omega_{u.k}^2 \int_0^1 R_k(s)^2 ds$$
$$F_{2,n}(k) \Rightarrow \omega_{u.k} \sup_{r\in[0,1]} |R_k(r)|,\; F_{3,n}(k) \Rightarrow \omega_{u.k} \sup_{r\in[0,1]} |R_k(r) - r \cdot R_k(1)|$$
$$\tilde{v}_n^2(k) \Rightarrow \omega_{u.k}^2 R_k(1)^2$$

and similarly for $F_{j,n}(p,k)$*, j = 1, 2, 3, and* $\tilde{v}_{n,p}^2(k)$ *with* $R_k(r)$ *replaced by* $R_{k,p}(r)$ *as has been defined in result (d) of Proposition 3.1. Also, under the alternative hypothesis of no cointegration, that is when* $|\alpha| < 1$ *in Assumption 2.1 with* $v = -1/2$*, then:*

$$(b)\, n^{-2} F_{1,n}(k) \Rightarrow \int_0^1 J_k(s)^2 ds$$
$$n^{-1} F_{2,n}(k) \Rightarrow \sup_{r\in[0,1]} |J_k(r)|,\; n^{-1} F_{3,n}(k) \Rightarrow \sup_{r\in[0,1]} |J_k(r) - r J_k(1)|$$
$$n^{-2} \tilde{v}_n^2(k) \Rightarrow J_k(1)^2$$

where

$$J_k(r) = J_u(r) - \mathbf{g}(r)' \left(\int_0^1 \mathbf{g}(s)\mathbf{g}(s)' ds \right)^{-1} \int_0^1 \mathbf{g}(s) J_u(s) ds$$

with $J_u(r) = \int_0^r B_u(s) ds$*, and similarly for* $n^{-2} F_{1,n}(p,k)$*,* $n^{-1} F_{j,n}(p,k)$*, j = 2, 3, and* $n^{-2} \tilde{v}_{n,p}^2(k)$*, with* $J_k(r)$ *replaced by* $J_{k,p}(r)$ *defined as*

$$J_{k,p}(r) = J_{u,p}(r) - \mathbf{g}_p(r)' \left(\int_0^1 \mathbf{g}_p(s)\mathbf{g}_p(s)' ds \right)^{-1} \int_0^1 \mathbf{g}_p(s) J_{u,p}(s) ds$$

where $J_{u,p}(r) = \int_0^r B_{u,p}(s) ds$.

Proof. See Appendix D.

Remark 4.1. As cited above, under the alternative of no cointegration, these test statistics are not consistent in the usual way because their limiting distributions are obtained without further normalization of the components in the numerator and denominator. However, these distributions differ from the null distributions in the sense that they are shifted to the left and more concentrated. This implies that a rejection of the null of cointegration against no cointegration is registered for small values of any of these test statistics, which means that this is a left tailed test that rejects the null hypothesis of cointegration for values of $\tilde{F}_{j,n}(k)$ smaller than the asymptotic critical value $c_{j,\alpha}(k)$ given by the αth-lower quantile of the asymptotic null distribution. From the results in part (a) of Proposition 3.1, it is evident that the asymptotic null distribution of all these test statistics are free of nuisance parameters and only depends on the combination of p and k in the case of using OLS detrended observations.

Tables 4.1 and 4.2 below present the critical values for the test statistics $\bar{F}_{j,n}(k)$ and $\bar{F}_{j,n}(p,k)$, for p = 0, 1, computed via direct simulation based on 20000 independent replications, with 2000 observations, and $\boldsymbol{\xi}_t = (u_t, \boldsymbol{\varepsilon}'_{k,t})' \sim$ iidN($\mathbf{0}_{k+1}$, $\mathbf{I}_{k+1}$), k = 1, ..., 5.

Remark 4.2. In the definition of all these test statistics, instead of using the simple normalization factor defined in (4.5) and (4.10) to eliminate the nuisance parameter $\omega^2_{u.k}$ in the fluctuation measures $F_{j,n}(p,k)$, we could consider the commonly used nonparametric kernel estimator, $\tilde{\omega}^2_n(m_n)$, based on the first differences of the IM-OLS residuals $\Delta\tilde{\zeta}_{t,p}(k)$, which is defined as

$$\begin{aligned}\tilde{\omega}^2_n(m_n) &= \sum_{j=-(n-1)}^{n-1} w(j/m_n)\left\{n^{-1}\sum_{t=|j|+1}^{n} \Delta\tilde{\zeta}_{t,p}(k)\Delta\tilde{\zeta}_{t-|j|,p}(k)\right\} \\ &= n^{-1}\sum_{t=1}^{n}\sum_{s=1}^{n} w\left(\frac{|t-s|}{m_n}\right)\Delta\tilde{\zeta}_{t,p}(k)\Delta\tilde{\zeta}_{s,p}(k)\end{aligned} \tag{4.11}$$

with bandwidth m_n and kernel function $w(\cdot)$. Irrespective of the choice of the kernel, the consistency of this estimator relies on the magnitude of the bandwidth parameter, m_n. Under stationarity and with $m_n = o_p(n^{1/2})$, which includes both the case of a sample-size dependent deterministic bandwidth choice and a data-dependent stochastic one, usually we must obtain the consistency result $\tilde{\omega}_n^2(m_n) \to^p \omega_{u.k}^2$, but this option requires the determination of a particular value for this parameter. In this setup, and from result (c) in Proposition 3.1, we have that the first difference of the IM-OLS residuals can be written as

$$\begin{aligned}\Delta\tilde{\zeta}_{t,p}(k) &= \Delta\zeta_{t,p} - n^{1-v}(n^{-3/2}\Delta\hat{\mathbf{S}}'_{kt,p}, n^{-1/2}\Delta\hat{\mathbf{T}}'_{kt,p})\begin{pmatrix} n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ n^{-1/2+v}(\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k)\end{pmatrix} \\ &= \Delta\zeta_{t,p} - n^{1-v}(n^{-3/2}\hat{\mathbf{X}}'_{kt,p}, n^{-1/2}\hat{\mathbf{Z}}'_{kt,p})\begin{pmatrix} n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ n^{-1/2+v}(\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k)\end{pmatrix}\end{aligned}$$

where $\Delta\zeta_{t,p} = z_{t,p} = u_{t,p} - \boldsymbol{\gamma}'_k\hat{\mathbf{Z}}_{kt,p}$, with $\hat{\mathbf{X}}_{kt,p} = \boldsymbol{\eta}_{kt,p}$, and $\hat{\mathbf{Z}}_{kt,p} = \boldsymbol{\varepsilon}_{kt,p}$, so that

$$\begin{aligned}\Delta\tilde{\zeta}_{t,p}(k) = u_{t,p} - \boldsymbol{\gamma}'_k\boldsymbol{\varepsilon}_{kt,p} - n^{1/2-v}n^{-1/2}(n^{-1/2}\boldsymbol{\eta}'_{kt,p})[n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k))] \\ - n^{1/2-v}\boldsymbol{\varepsilon}'_{kt,p}[n^{-1/2+v}(\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k)]\end{aligned}$$

Under the assumption of cointegration we have that

$$\begin{aligned}\Delta\tilde{\zeta}_{t,p}(k) &= u_t - \boldsymbol{\gamma}'_k\boldsymbol{\varepsilon}_{k,t} - \boldsymbol{\varepsilon}'_{k,t}(\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k) + O_p(n^{-1/2}) \\ &= (z_t, \boldsymbol{\varepsilon}'_{k,t})\begin{pmatrix} 1 \\ -(\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k)\end{pmatrix} + O_p(n^{-1/2})\end{aligned}$$

given that $u_{t,p} = u_t + O_p(n^{-1/2})$, and $\boldsymbol{\varepsilon}_{kt,p} = \boldsymbol{\varepsilon}_{k,t} + O_p(n^{-1/2})$, with z_t and $\boldsymbol{\varepsilon}_{k,t}$ zero-mean stationary processes that are asymptotically uncorrelated by construction, so that the long-run covariance matrix of $(z_t, \boldsymbol{\varepsilon}'_{k,t})'$ is $diag(\omega_{u.k}^2, \boldsymbol{\Omega}_{kk})$. With these, and using the result (b) in Proposition 3.1 above in more compact form as

$$\begin{pmatrix} n(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ \tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k\end{pmatrix} \Rightarrow \omega_{u.k}\boldsymbol{\Pi}^{-1}\begin{pmatrix}\mathbf{d}_{\beta,k} \\ \mathbf{d}_{\gamma,k}\end{pmatrix} = \omega_{u.k}\begin{pmatrix}\boldsymbol{\Omega}_{kk}^{-1/2}\mathbf{d}_{\beta,k} \\ \boldsymbol{\Omega}_{kk}^{-1/2}\mathbf{d}_{\gamma,k}\end{pmatrix}$$

then we get $\tilde{\omega}_n^2(m_n) \Rightarrow \omega_{u.k}^2(1 + \mathbf{d}'_{\gamma,k}\mathbf{d}_{\gamma,k})$,[7] which is a random limit and is given by the

[7] For a more detailed demonstration of this result, see page 32 in Vogelsang and Wagner (2011).

random vector $\mathbf{d}_{\gamma,k}$ determining the limiting null distribution of $\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k$. In this case, using $\tilde{\omega}_n^2(m_n)$ with some simple rule for determining the bandwidth under stationarity, we conjecture that this will produce consistent test statistics. Formally, given that under the assumption of no cointegration we have

$$\begin{aligned} n^{-1/2}\Delta\tilde{\zeta}_{t,p}(k) &= n^{-1/2}u_{t,p} - (n^{-1/2}\boldsymbol{\eta}'_{kt,p})(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ &\quad - \sqrt{n}\boldsymbol{\varepsilon}'_{k,t}[n^{-1}(\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k)] + O_p(1) = O_p(\sqrt{n}) \end{aligned}$$

and thus $\Delta\tilde{\zeta}_{t,p}(k) = O_p(n)$ using (C.6) in Appendix C, then

$$n^{-2}\tilde{\omega}_n^2(m_n) = (\tilde{\boldsymbol{\gamma}}'_{k,n}/n)\sum_{j=-(n-1)}^{n-1} w(j/m_n)\left\{n^{-1}\sum_{t=|j|+1}^{n} \boldsymbol{\varepsilon}_{k,t}\boldsymbol{\varepsilon}'_{k,t-|j|}\right\}(\tilde{\boldsymbol{\gamma}}_{k,n}/n) + o_p(1)$$

with a well defined stochastic limit, so that $\tilde{\omega}_n^2(m_n) = O_p(n^2)$. Alternatively, and following the idea developed by Kiefer and Vogelsang (2005), and further analized by Sun, Phillips and Jin (2008), we could consider the so called fixed-b estimation theory of a long-run variance based on a bandwidth that is simply proportional to the sample size as $m_n = b \cdot n$, with $b \in (0,1]$. The results in this case were extended by Vogelsang and Wagner (2011) to models with nonstationary regressors, but the asymptotics are relatively more complex and no treated here. A particular case, that can be treated without any additional development, is when $b = 1$ so that the bandwidth is set equal to the sample size, $m_n = n$. By using Lemma 1 in Cai and Shintani (2006) for the Bartlett kernel, $w(x) = 1-|x|$, for $|x| \leq 1$, we can write (4.11) as follows

$$\begin{aligned} \tilde{\omega}_n^2(n) = n^{1-2v}\Bigg(2n^{-1}\Bigg\{&\sum_{t=1}^{n}(n^{-(1-v)}\tilde{\zeta}_{t,p}(k))^2 \\ &-(n^{-(1-v)}\tilde{\zeta}_{n,p}(k))\sum_{t=1}^{n}(n^{-(1-v)}\tilde{\zeta}_{t,p}(k))\Bigg\} + (n^{-(1-v)}\tilde{\zeta}_{n,p}(k))^2\Bigg) \end{aligned} \tag{4.12}$$

whose asymptotic distribution is proportional to $\omega_{u.k}^2$ under the cointegration assumption and, as for the simple element $\tilde{v}_{n,p}^2(k)$, is of the same order of magnitude as the fluctuation measures (4.8) and (4.9), both under cointegration and no cointegration, resulting in inconsistent test statistics. Although all these other options seems to produce inconsistent test statistics, we explore their use in future research.

Table 4.1 Asymptotic lower critical values for the cointegration test based on the fluctuation test statistic $\tilde{F}_{j,n}(k)$, j = 1, 2, 3

	Significance level, α	$k=1$	2	3	4	5
Test statistic $\tilde{F}_{1,n}(k)$	0.01	0.0874	0.0834	0.0847	0.0823	0.0811
	0.025	0.1101	0.1077	0.1076	0.1083	0.1051
	0.05	0.1383	0.1363	0.1357	0.1376	0.1352
	0.1	0.1863	0.1871	0.1875	0.1917	0.1912
Test statistic $\tilde{F}_{2,n}(k)$	0.01	1.0000	1.0000	1.0000	1.0000	1.0000
	0.025	1.0074	1.0143	1.0175	1.0289	1.0331
	0.05	1.0393	1.0642	1.0885	1.1079	1.1251
	0.1	1.1225	1.1964	1.2458	1.2926	1.3197
Test statistic $\tilde{F}_{3,n}(k)$	0.01	0.9382	1.3522	1.4484	1.4877	1.5162
	0.025	1.1235	1.4553	1.5510	1.5889	1.6245
	0.05	1.2668	1.5605	1.6437	1.7080	1.7391
	0.1	1.4580	1.7169	1.8110	1.8701	1.9079

Table 4.2 Asymptotic lower critical values for the cointegration test based on the fluctuation test statistic $\tilde{F}_{j,n}(p,k)$, j = 1, 2, 3

		Demeaned case, $p=0$					Demeaned and detrended case, $p=1$				
	Significance level, α	$k=1$	2	3	4	5	$k=1$	2	3	4	5
$\tilde{F}_{1,n}(p,k)$	0.01	0.233	0.166	0.138	0.123	0.117	0.304	0.217	0.180	0.157	0.143
	0.025	0.344	0.223	0.184	0.165	0.154	0.470	0.307	0.249	0.214	0.197
	0.05	0.507	0.312	0.248	0.219	0.204	0.718	0.436	0.345	0.288	0.258
	0.1	0.827	0.466	0.370	0.317	0.292	1.237	0.686	0.519	0.431	0.381
$\tilde{F}_{2,n}(p,k)$	0.01	1.266	1.139	1.087	1.061	1.067	1.493	1.321	1.249	1.183	1.158
	0.025	1.538	1.323	1.240	1.196	1.184	1.872	1.572	1.463	1.361	1.342
	0.05	1.832	1.545	1.433	1.378	1.356	2.321	1.885	1.708	1.609	1.553
	0.1	2.371	1.904	1.751	1.657	1.620	3.077	2.380	2.121	1.975	1.892
$\tilde{F}_{3,n}(p,k)$	0.01	1.613	1.627	1.656	1.642	1.666	1.870	1.799	1.775	1.765	1.758
	0.025	1.870	1.819	1.809	1.792	1.792	2.220	2.053	1.985	1.956	1.925
	0.05	2.181	2.021	1.978	1.961	1.943	2.648	2.308	2.213	2.149	2.109
	0.1	2.655	2.340	2.254	2.205	2.177	3.330	2.747	2.576	2.474	2.412

Next, in order to evaluate the power of these test statistics we use a local-to-unity approach to cointegration in finite samples where the error term in the cointegrating regression equation follows the AR(1) process $u_t = \alpha_n u_{t-1} + \upsilon_t$, with $\alpha_n = 1 - c/n$, $c \geq 0$ as in Phillips (1987), which gives the following result.

Corolary 4.2. *Under the local-to-unity approach to the null of cointegration and*

Assumption 2.1, then we have that $n^{-1/2}u_{[nr]} \Rightarrow B_c(r) = \int_0^r e^{(r-s)c} dB_\upsilon(s)$, *and*

$$n^{-3/2}\tilde{\zeta}_{[nr],p}(k) \Rightarrow J_{c,k,p}(r)$$

where $J_{c,k,p}(r)$ *is as* $J_{k,p}(r)$ *in part (b) of Proposition 4.1, with* $J_{u,p}(r)$ *replaced by* $J_{c,p}(r) = \int_0^r B_{c,p}(s)ds$, *and* $B_{c,p}(s)$ *the detrended Ornstein-Uhlenbeck process* $B_c(s)$.

Next Table 4.3 shows the power results for sample sizes n = 100 and 500 computed by simulation with 5000 replications for values of c = 0, 1, 2.5, 5 and 10 for the fluctuation-type test statistic $\overline{F}_{3,n}(p,k)$, with p = 0, 1 and k = 1, …, 5.

Table 4.3 Finite-sample power of the test statistic $\overline{F}_{3,n}(p,k)$, p = 0, 1, under the local-to-unity approach to stationarity (cointegration) at the 5% nominal level

		Case p = 0, $\overline{F}_{3,n}(0,k)$		Case p = 1, $\overline{F}_{3,n}(1,k)$	
Sample size		n = 100	500	n = 100	500
c = 0	k = 1	0.1770	0.1806	0.1492	0.1634
	2	0.1842	0.2016	0.1512	0.1860
	3	0.1830	0.2134	0.1556	0.1836
	4	0.1872	0.2044	0.1678	0.1846
	5	0.1954	0.2034	0.1554	0.1798
c = 1	k = 1	0.1658	0.1992	0.1298	0.1698
	2	0.1696	0.2004	0.1526	0.1920
	3	0.1926	0.2174	0.1562	0.1774
	4	0.1768	0.2262	0.1664	0.1950
	5	0.1724	0.2100	0.1584	0.2016
c = 2.5	k = 1	0.1684	0.1762	0.1580	0.1720
	2	0.1738	0.2018	0.1500	0.1764
	3	0.1890	0.1876	0.1578	0.1952
	4	0.1696	0.1820	0.1600	0.1874
	5	0.1792	0.1882	0.1620	0.1948
c = 5	k = 1	0.1502	0.1614	0.1436	0.1606
	2	0.1528	0.1766	0.1510	0.1696
	3	0.1618	0.1850	0.1656	0.1556
	4	0.1612	0.1722	0.1588	0.1704
	5	0.1624	0.1826	0.1604	0.1852
c = 10	k = 1	0.1200	0.1352	0.1244	0.1262
	2	0.1500	0.1442	0.1436	0.1560
	3	0.1488	0.1628	0.1420	0.1526
	4	0.1534	0.1750	0.1494	0.1614
	5	0.1432	0.1724	0.1408	0.1696

The first remarkable evidence is that of inconsistency of the proposed test statistic, and the very low power displayed irrespective of the value of c, which indicates the need of a more deeply investigation of these testing procedures.

5. Conclusions and some extensions

The present paper is devoted to the analysis of the asymptotically efficient estimation of a linear static cointegrating regression model by making use of a new recently proposed estimation method by Vogelsang and Wagner (2005), the so-called integrated modified OLS estimator (IM-OLS) that has the main advantage that does not require the choice of any tuning parameter, when we deal with deterministically trending integrated regressors. We show that this method must be modified to correctly accommodate the structure of the deterministic component of the regressors and to avoid possible inconsistencies in the estimation results. As a byproduct of these results, we propose the use of the IM-OLS residuals to build some new simple statistics to testing the null hypothesis of cointegration against the alternative of no cointegration. While the main component of these new test statistics seems to work well in detecting excessive fluctuations in the residual sequence under no cointegration, it is not yet clear how to obtain pivotal test statistics free of nuisance parameters and consistent tests given the difficulties in obtaining a proper estimator of a long-run variance. This central question will be studied in future work, as well as the consideration of more complex deterministic components and their treatment in the context of the IM-OLS estimation.

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Appendix

A. Proof of Proposition 2.1. By OLS detrending of the observed processes Y_t and $\mathbf{X}_{k,t}$, as defined by (2.1) and (2.2), we have that

$$\begin{pmatrix} \hat{Y}_{t,p} \\ \hat{\mathbf{X}}_{kt,p} \end{pmatrix} = \begin{pmatrix} Y_t \\ \mathbf{X}_{k,t} \end{pmatrix} - \sum_{j=1}^{n} \begin{pmatrix} Y_j \\ \mathbf{X}_{k,j} \end{pmatrix} \boldsymbol{\tau}'_{p,j} \cdot \mathbf{Q}_{n,pp}^{-1} \boldsymbol{\tau}_{p,t} \qquad t = 1, \ldots, n$$

Each of the components above can be decomposed as

$$\eta_{it,p} + \boldsymbol{\alpha}'_{i,p_i} (\boldsymbol{\tau}_{p_i,t} - \textstyle\sum_{j=1}^{n} \boldsymbol{\tau}_{p_i,j} \boldsymbol{\tau}'_{p,j} \mathbf{Q}_{n,pp}^{-1} \boldsymbol{\tau}_{p,t}) \qquad i = 0,\ 1,\ \ldots,\ k,$$

where $\eta_{it,p} = \eta_{i,t} - \sum_{j=1}^{n} \eta_{i,j} \boldsymbol{\tau}'_{p,j} \mathbf{Q}_{n,pp}^{-1} \boldsymbol{\tau}_{p,t}$, and

$$\begin{aligned}
\boldsymbol{\tau}_{p_i,t} - \sum_{j=1}^{n} \boldsymbol{\tau}_{p_i,j} \boldsymbol{\tau}'_{p,j} \mathbf{Q}_{n,pp}^{-1} \boldsymbol{\tau}_{p,t} &= \boldsymbol{\tau}_{p_i,t} - \sum_{j=1}^{n} \boldsymbol{\tau}_{p_i,j} (\boldsymbol{\tau}'_{p_i,j} : \boldsymbol{\tau}'_{p-p_i,j}) \mathbf{Q}_{n,pp}^{-1} \begin{pmatrix} \boldsymbol{\tau}_{p_i,t} \\ \boldsymbol{\tau}_{p-p_i,t} \end{pmatrix} \\
&= \boldsymbol{\tau}_{p_i,t} - (\mathbf{Q}_{n,p_i p_i} : \mathbf{Q}_{n,p_i(p-p_i)}) \begin{pmatrix} \mathbf{Q}_{n,p_i p_i} & \mathbf{Q}_{n,p_i(p-p_i)} \\ \mathbf{Q}'_{n,p_i(p-p_i)} & \mathbf{Q}_{n,(p-p_i)(p-p_i)} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\tau}_{p_i,t} \\ \boldsymbol{\tau}_{p-p_i,t} \end{pmatrix} \\
&= \boldsymbol{\tau}_{p_i,t} - (\mathbf{I}_{p_i+1,p_i+1} : \mathbf{0}_{p_i+1,p-p_i}) \begin{pmatrix} \boldsymbol{\tau}_{p_i,t} \\ \boldsymbol{\tau}_{p-p_i,t} \end{pmatrix} = \mathbf{0}_{p_i+1}
\end{aligned}$$

given the block structure for the inverse of $\mathbf{Q}_{n,pp}$, when $p_i < p$ for all $i = 0, 1, \ldots, k$. Obviously, the same result directly holds when $p_i = p$, while that if any $p_i > p$, then we have $\boldsymbol{\alpha}'_{i,p_i} (\boldsymbol{\tau}_{p_i,t} - \sum_{j=1}^{n} \boldsymbol{\tau}_{p_i,j} \boldsymbol{\tau}'_{p,j} \mathbf{Q}_{n,pp}^{-1} \boldsymbol{\tau}_{p,t}) = \boldsymbol{\alpha}'_{i,p_i-p} (\boldsymbol{\tau}_{p_i-p,t} - \mathbf{Q}_{n,(p_i-p)p} \mathbf{Q}_{n,pp}^{-1} \boldsymbol{\tau}_{p,t})$, which does not vanish and it is of order $O(n^{p_i})$.

■

B. Proof of Proposition 2.2. First, given that we can write

$$\left(\sum_{t=1}^{n}\begin{pmatrix}\boldsymbol{\tau}_{p,tn}\\ \boldsymbol{\eta}_{k,tn}\end{pmatrix}(\boldsymbol{\tau}'_{p,tn},\boldsymbol{\eta}'_{k,tn})\right)^{-1}=\begin{pmatrix}\mathbf{Q}_{n,pp} & \mathbf{Q}_{n,pk}\\ \mathbf{Q}'_{n,pk} & \mathbf{Q}_{n,kk}\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}\mathbf{M}^{-1}_{n,pp} & -\mathbf{M}^{-1}_{n,pp}\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\\ -\mathbf{M}^{-1}_{n,kk}\mathbf{Q}'_{n,pk}\mathbf{Q}^{-1}_{n,pp} & \mathbf{M}^{-1}_{n,kk}\end{pmatrix}$$

then, using (2.11) we have that

$$\begin{pmatrix}\mathbf{M}^{-1}_{n,pp} & -\mathbf{M}^{-1}_{n,pp}\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\\ -\mathbf{M}^{-1}_{n,kk}\mathbf{Q}'_{n,pk}\mathbf{Q}^{-1}_{n,pp} & \mathbf{M}^{-1}_{n,kk}\end{pmatrix}\sum_{t=1}^{n}\begin{pmatrix}\boldsymbol{\tau}_{p,tn}\\ \boldsymbol{\eta}_{k,tn}\end{pmatrix}\Delta\mathbf{X}'_{k,t}$$
$$=\begin{pmatrix}\mathbf{M}^{-1}_{n,pp} & -\mathbf{M}^{-1}_{n,pp}\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\\ -\mathbf{M}^{-1}_{n,kk}\mathbf{Q}'_{n,pk}\mathbf{Q}^{-1}_{n,pp} & \mathbf{M}^{-1}_{n,kk}\end{pmatrix}\left\{\begin{pmatrix}\mathbf{Q}_{n,pp}\boldsymbol{\Gamma}^{-1}_{p,n}\boldsymbol{\Phi}'_{k,p}\\ \mathbf{Q}'_{n,pk}\boldsymbol{\Gamma}^{-1}_{p,n}\boldsymbol{\Phi}'_{k,p}\end{pmatrix}+\sum_{t=1}^{n}\begin{pmatrix}\boldsymbol{\tau}_{p,tn}\\ \boldsymbol{\eta}_{k,tn}\end{pmatrix}\boldsymbol{\varepsilon}'_{k,t}\right\}$$
$$=\begin{pmatrix}\boldsymbol{\Gamma}^{-1}_{p,n}\boldsymbol{\Phi}'_{k,p}\\ \mathbf{0}_{k,k}\end{pmatrix}+\begin{pmatrix}\mathbf{M}^{-1}_{n,pp}\left\{\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{k,t}\right\}\\ \mathbf{M}^{-1}_{n,kk}\left\{\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}'_{n,pk}\mathbf{Q}^{-1}_{n,pp}\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}\right\}\end{pmatrix}$$

and

$$\begin{pmatrix}\mathbf{M}^{-1}_{n,pp} & -\mathbf{M}^{-1}_{n,pp}\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\\ -\mathbf{M}^{-1}_{n,kk}\mathbf{Q}'_{n,pk}\mathbf{Q}^{-1}_{n,pp} & \mathbf{M}^{-1}_{n,kk}\end{pmatrix}\mathbf{W}^{-1}_{n}\begin{pmatrix}\mathbf{0}_{p+1}\\ n\Delta^{+}_{ku}\end{pmatrix}=\sqrt{n}\begin{pmatrix}-\mathbf{M}^{-1}_{n,pp}\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\Delta^{+}_{ku}\\ \mathbf{M}^{-1}_{n,kk}\Delta^{+}_{ku}\end{pmatrix}$$

with $\mathbf{W}_n$ given in (2.4). Taking these results together we get (2.12). Second, given the sequence of FM-OLS residuals, defined by $\hat{u}^{+}_{t,p}(k)=Y^{+}_{t}-(\boldsymbol{\tau}'_{p,t},\mathbf{X}'_{k,t})(\hat{\boldsymbol{\alpha}}'^{+}_{p,n},\hat{\boldsymbol{\beta}}'^{+}_{k,n})'$, with $Y^{+}_{t}=Y_t-(\boldsymbol{\tau}'_{p,tn}\boldsymbol{\Gamma}^{-1}_{p,n}\boldsymbol{\Phi}'_{k,p}+\boldsymbol{\varepsilon}'_{k,t})\boldsymbol{\gamma}_k$, can be written as

$$\hat{u}^{+}_{t,p}(k)=\hat{u}_{t,p}(k)-\boldsymbol{\varepsilon}'_{k,t}\boldsymbol{\gamma}_k$$
$$+\boldsymbol{\tau}'_{p,tn}\mathbf{M}^{-1}_{n,pp}\left(\left\{\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{k,t}\right\}\boldsymbol{\gamma}_k-\sqrt{n}\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\Delta^{+}_{ku}\right)$$
$$+\boldsymbol{\eta}'_{k,tn}\mathbf{M}^{-1}_{n,kk}\left(\left\{\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}'_{n,pk}\mathbf{Q}^{-1}_{n,pp}\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}\right\}\boldsymbol{\gamma}_k+\sqrt{n}\Delta^{+}_{ku}\right)$$

or, in more compact form, as in (2.13) when using

$$\mathbf{M}^{-1}_{n,pp}\left\{\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}_{n,pk}\mathbf{Q}^{-1}_{n,kk}\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{k,t}\right\}=\mathbf{Q}^{-1}_{n,pp}\left\{\sum_{t=1}^{n}\boldsymbol{\tau}_{p,tn}\boldsymbol{\varepsilon}'_{k,t}-\mathbf{Q}_{n,pk}\mathbf{M}^{-1}_{n,kk}\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{kt,p}\right\}$$

and $\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}\boldsymbol{\varepsilon}'_{kt,p}=\sum_{t=1}^{n}\boldsymbol{\eta}_{k,tn}(\boldsymbol{\varepsilon}'_{k,t}-\boldsymbol{\tau}'_{p,tn}\mathbf{Q}^{-1}_{n,pp}\sum_{t=1}^{n}\boldsymbol{\tau}_{p,jn}\boldsymbol{\varepsilon}'_{k,j})$.

■

C. Proof of Proposition 3.1(a, b). Partial summing from (2.8) gives

$$\hat{S}_{t,p}=\boldsymbol{\beta}'_k\hat{\mathbf{S}}_{kt,p}+U_{t,p},t=1,...,n \qquad \text{(C.1)}$$

so that

$$\begin{pmatrix}\tilde{\boldsymbol{\beta}}_{k,n}\\ \tilde{\boldsymbol{\gamma}}_{k,n}\end{pmatrix}=\begin{pmatrix}\boldsymbol{\beta}_k\\ \mathbf{0}_k\end{pmatrix}+n^{(1-v)}(\mathbf{W}'_n)^{-1}\left((1/n)\sum_{t=1}^{n}\begin{pmatrix}n^{-3/2}\hat{\mathbf{S}}_{kt,p}\\ n^{-1/2}\hat{\mathbf{T}}_{kt,p}\end{pmatrix}(n^{-3/2}\hat{\mathbf{S}}'_{kt,p},n^{-1/2}\hat{\mathbf{T}}'_{kt,p})\right)^{-1}$$
$$\times(1/n)\sum_{t=1}^{n}\begin{pmatrix}n^{-3/2}\hat{\mathbf{S}}_{kt,p}\\ n^{-1/2}\hat{\mathbf{T}}_{kt,p}\end{pmatrix}n^{-(1-v)}U_{t,p}$$

and thus

$$n^{-(1-v)}\mathbf{W}_n'\begin{pmatrix}\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k\\ \tilde{\boldsymbol{\gamma}}_{k,n}\end{pmatrix}=\begin{pmatrix}n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k)\\ n^{-1/2+v}\tilde{\boldsymbol{\gamma}}_{k,n}\end{pmatrix}$$

$$=\left((1/n)\sum_{t=1}^{n}\begin{pmatrix}n^{-3/2}\hat{\mathbf{S}}_{kt,p}\\ n^{-1/2}\hat{\mathbf{T}}_{kt,p}\end{pmatrix}(n^{-3/2}\hat{\mathbf{S}}_{kt,p}',n^{-1/2}\hat{\mathbf{T}}_{kt,p}')\right)^{-1}\left\{(1/n)\sum_{t=1}^{n}\begin{pmatrix}n^{-3/2}\hat{\mathbf{S}}_{kt,p}\\ n^{-1/2}\hat{\mathbf{T}}_{kt,p}\end{pmatrix}n^{-(1-v)}U_{t,p}\right\} \tag{C.2}$$

Making use of the convergence results in (2.14), (2.15) and (3.11), then under the cointegration assumption, that is when $v = 1/2$, we have that

$$\begin{pmatrix}n(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k)\\ \tilde{\boldsymbol{\gamma}}_{k,n}\end{pmatrix}\Rightarrow\left(\int_0^1\mathbf{g}_p(r)\mathbf{g}_p(r)'dr\right)^{-1}\int_0^1\mathbf{g}_p(r)V_{u,p}(r)dr$$

$$=\left(\int_0^1\mathbf{g}_p(r)\mathbf{g}_p(r)'dr\right)^{-1}\left\{\int_0^1\mathbf{g}_p(r)V_{u.k,p}(r)dr+\int_0^1\mathbf{g}_p(r)\mathbf{V}_{k,p}(r)'dr\,\boldsymbol{\gamma}_k\right\} \tag{C.3}$$

where the last two terms are based on the decomposition in (2.17). For the last term above, as in Vogelsang and Wagner (2011) (equation (43), page 27), we can write

$$\int_0^1\mathbf{g}_p(r)\mathbf{V}_{k,p}(r)'dr\,\boldsymbol{\gamma}_k=\int_0^1\mathbf{g}_p(r)\left\{\mathbf{g}_p(r)'\begin{pmatrix}\mathbf{0}_{k,k}\\ \mathbf{I}_{k,k}\end{pmatrix}\right\}dr\,\boldsymbol{\gamma}_k=\int_0^1\mathbf{g}_p(r)\mathbf{g}_p(r)'dr\begin{pmatrix}\mathbf{0}_k\\ \boldsymbol{\gamma}_k\end{pmatrix} \tag{C.4}$$

so that

$$\begin{pmatrix}n(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k)\\ \tilde{\boldsymbol{\gamma}}_{k,n}-\boldsymbol{\gamma}_k\end{pmatrix}\Rightarrow\boldsymbol{\Pi}^{-1}\left(\int_0^1\overline{\mathbf{g}}_p(r)\overline{\mathbf{g}}_p(r)'dr\right)^{-1}\int_0^1\overline{\mathbf{g}}_p(r)V_{u.k,p}(r)dr$$

or, equivalently,

$$\begin{pmatrix}n(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k)\\ \tilde{\boldsymbol{\gamma}}_{k,n}-\boldsymbol{\gamma}_k\end{pmatrix}\Rightarrow\boldsymbol{\Pi}^{-1}\left(\int_0^1\overline{\mathbf{g}}_p(r)\overline{\mathbf{g}}_p(r)'dr\right)^{-1}\int_0^1[\mathbf{G}_p(1)-\mathbf{G}_p(r)]dV_{u.k,p}(r)$$

where the last equality comes from defining $\mathbf{G}_p(r)=\int_0^r\mathbf{g}_p(s)ds=\boldsymbol{\Pi}\int_0^r\overline{\mathbf{g}}_p(s)ds$, with $\boldsymbol{\Pi}=diag(\boldsymbol{\Omega}_{k,k}^{1/2},\boldsymbol{\Omega}_{k,k}^{1/2})$, and $\mathbf{g}_p(r)=\boldsymbol{\Pi}\cdot\overline{\mathbf{g}}_p(r)$. Also, by defining $V_{u.k,p}(r)=\omega_{u.k}^2\cdot W_{u.k,p}(r)$, with $W_{u.k,p}(r)=BM(b_p(r))$, then we have

$$\begin{pmatrix}n(\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k)\\ \tilde{\boldsymbol{\gamma}}_{k,n}-\boldsymbol{\gamma}_k\end{pmatrix}\Rightarrow\omega_{u.k}\boldsymbol{\Pi}^{-1}\left(\int_0^1\overline{\mathbf{g}}_p(r)\overline{\mathbf{g}}_p(r)'dr\right)^{-1}\int_0^1[\mathbf{G}_p(1)-\mathbf{G}_p(r)]dW_{u.k,p}(r) \tag{C.5}$$

As in equation (24) in Vogelsang and Wagner (2011), conditional on $\mathbf{B}_k(r)$, the above limiting distribution (C.5) is $N(\mathbf{0}_{2k},\boldsymbol{\Theta}_{2k})$, with $\boldsymbol{\Theta}_{2k}$ a well defined conditional asymptotic stochastic covariance matrix. Under no cointegration, that is, with $v = -1/2$ and nonstationarity of the error sequence u_t, then we have

$$\begin{pmatrix}\tilde{\boldsymbol{\beta}}_{k,n}-\boldsymbol{\beta}_k\\ n^{-1}\tilde{\boldsymbol{\gamma}}_{k,n}\end{pmatrix}\Rightarrow\left(\int_0^1\mathbf{g}_p(r)\mathbf{g}_p(r)'dr\right)^{-1}\int_0^1\mathbf{g}_p(r)J_{u,p}(r)dr \tag{C.6}$$

where $J_{u,p}(r)=\int_0^r B_{u,p}(s)ds$. As can be seen from (C.5) and (C.6), the convergence rates for the IM-OLS estimator of $\boldsymbol{\beta}_k$ are the same as when using OLS or any of the asymptotically equivalent and efficient estimation methods.

Proof of Proposition 3.1(c, d). Given the IM-OLS residual sequence in (3.14), the IM cointegrating regression equation in (3.9) and (C.2), we can write $\tilde{\zeta}_{t,p}(k)$ as

$$\tilde{\zeta}_{t,p}(k) = \zeta_{t,p} - n^{1-v}(n^{-3/2}\hat{\mathbf{S}}'_{kt,p}, n^{-1/2}\hat{\mathbf{T}}'_{kt,p})\begin{pmatrix} n^{1/2+v}(\tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k) \\ n^{-1/2+v}(\tilde{\boldsymbol{\gamma}}_{k,n} - \boldsymbol{\gamma}_k) \end{pmatrix} \quad t = 1,\ldots,n$$

Under the cointegration assumption, making use of (3.11), (3.12) and the weak convergence of the IM-OLS estimators of $\boldsymbol{\beta}_k$ and $\boldsymbol{\gamma}_k$, the result (d) then follows by the continuous mapping theorem.

■

D. Proof of Proposition 4.1(a). It follows directly from the results in (b) and (d) from Proposition 3.1 and the continuous mapping theorem.

Proof of Proposition 4.1(b). From result (c) in Proposition 3.1 with $v = -1/2$ we have

$$n^{-3/2}\tilde{\zeta}_{t,p}(k) = n^{-3/2}\zeta_{t,p} - (n^{-3/2}\hat{\mathbf{S}}'_{kt,p}, n^{-1/2}\hat{\mathbf{T}}'_{kt,p})\begin{pmatrix} \tilde{\boldsymbol{\beta}}_{k,n} - \boldsymbol{\beta}_k \\ n^{-1}\tilde{\boldsymbol{\gamma}}_{k,n} \end{pmatrix}$$

where

$$n^{-3/2}\zeta_{t,p} = n^{-3/2}U_{t,p} - \boldsymbol{\gamma}'_k n^{-1}(n^{-1/2}\hat{\mathbf{T}}_{kt,p}) = n^{-3/2}U_{t,p} + O_p(n^{-1})$$

so that, using (C.6) above and the continuous mapping theorem we have that

$$n^{-3/2}\tilde{\zeta}_{t,p}(k) \Rightarrow J_{k,p}(r) = J_{u,p}(r) - \mathbf{g}_p(r)'\left(\int_0^1 \mathbf{g}_p(s)\mathbf{g}_p(s)'ds\right)^{-1}\int_0^1 \mathbf{g}_p(s)J_{u,p}(s)ds$$

with $J_{u,p}(r) = \int_0^r B_{u,p}(s)ds$ as in (2.14), which gives the indicated results.

■