School Choice:
The Case for the Boston Mechanism

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Abstract

I study school assignment problems, focusing on two popular mechanisms: the Boston Mechanism (BM) and Deferred Acceptance (DA). The former has been criticized regarding both efficiency and fairness, particularly its treatment of naïve (non-strategic) students. The latter has been suggested in its place, and has already replaced the former in several cities. The formal critique of BM and support of DA were founded on the assumption of strict priorities, i.e., schools rank every child so that there are as many priority classes as there are students. In almost all cities where these mechanisms are applied, however, the actual number of priority classes (e.g., walking-distance and sibling in school) that may be used is orders of magnitude smaller than the number of students, and tie-breaking lotteries are needed. Approximating this case by assuming only one priority class, I show that BM outperforms DA according to several *ex ante* efficiency criteria. DA performs very poorly if all students share identical ordinal preferences over schools. Simulations show that these analytical results extend to more realistic cases. Finally, I suggest a simple modification to BM, which, according to simulations, protects naïve students while largely preserving its efficiency properties.

Keywords: School choice, efficiency, school priorities, naïve students.

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1 Introduction

The choice among allocation mechanisms has considerable consequences on efficiency outcomes.[1] In school choice, selecting an adequate assignment mechanism is particularly relevant due to the importance of providing children with the kind of schooling parents desire for them. There is an ongoing debate on what school assignment mechanism accomplishes with a more satisfactory set of properties regarding simplicity, efficiency and fairness. Most of this debate has centered its attention on two popular methods: the Boston Mechanism (BM) and Deferred Acceptance (DA).

Criticisms against BM have increased in recent years. BM is not strategy-proof. Consequently, parents are forced to play a cumbersome game. This raises concerns about efficiency (the strategic interaction may have pervasive effects, and agents may lack sufficient information to design a best response) and fairness (truth-tellers can do poorly).[2] DA has been suggested in its place, since it is strategy-proof and it relies only on ordinal preferences.

DA has replaced BM in several major districts such as Boston, MA and Seattle, WA. Nevertheless, variations of BM are still applied in other districts, for instance Denver, CO, Minneapolis, MN, and Cambridge, MA.[3] The purpose of the present paper is to note some positive aspects of BM that may temper the judgement against it, and to provide some rationale to its persistence.

A school usually has to prioritize over students if it cannot provide a slot to each of them. The formal analysis which has concluded against BM has assumed that each school ranks every child so that there are as many priority classes as there are students. This eliminates any uncertainty about which students would be accepted in the event the school were overdemanded. I refer to this context as the strict-priority case.

However, strict priorities constitute a strong assumption in school choice. In Boston, each school

[1] This has been documented in spectrum auctions and in the contracting-out of publicly provided services, among other examples.

[2] In BM, Abdulkadiroğlu, Pathak, Roth and Sönmez (2006) report that some parents behave strategically while some others are naïve (truth-tellers). That is, some parents are in an advantageous position to obtain necessary information and to calculate optimal strategies. Following this concern about fairness, Pathak and Sönmez (2008) show that perfectly strategic parents have an advantage over naïve (truth-telling) parents that can be accounted for as if all schools gave priority to strategic parents.

[3] Other systems are also in use. For instance, Providence, RI, allows parents to rank two of the district's schools. Each school reserves 75% slots for walking-zone students (those who live close enough to the school) who point at it as first choice, 25% for non-walking-zone students who rank the school first, and 5% for walking-zone students who rank it in second position. In each segment, priority is given to students with a sibling attending the school, and a lottery serves to break ties.

prioritizes over students according to four categories: 1) sibling's presence plus walking zone (residence close enough to school), 2) sibling only, 3) walking zone only, and 4) none. These four priority levels are also used in many other districts, sometimes jointly with readjustments that balance various socio-demographic variables across schools. Each of the three zones the Boston Public School authority divides the district into had an average of more than 1,100 elementary school applicants in the 2001-2002 assignment[4] (higher numbers are found in middle schools). Cambridge processed 431 applications in its most recent kindergarten assignment. Obviously, one cannot generate strict priorities over so many students with only four categories.

The few-categories reality of school choice is commonly referred to as "weak priorities." The no-priority assumption is an alternative, analytically tractable case that proxies such a scenario. Under no priorities, all students belong to the same priority class. I compare BM and DA based on this assumption. In my calculations, I restrict attention to large economies with a continuum of students. I find that BM obtains the same standard efficiency properties as DA, plus a set of additional *ex ante* efficiency properties. I also present simulations suggesting that BM is *ex ante* better than DA in the utilitarian sense even after the introduction of naïve students and weak priorities. I explain my results in more detail.

In both BM and DA, each student (or student's parents) is requested to report a complete ranking of the schools. The assignment procedure follows then several rounds. In each round, each student's application is considered at the highest-ranked school among the ones that have not rejected her yet. Rejections from a school occur when there are more applicants than slots the school offers. Acceptances and rejections are determined by school priorities and tie-breaking lotteries when needed. The difference between BM and DA is that an accepted student in BM keeps her assigned slot and is removed from further rounds. That is, acceptances are definite. In DA, an accepted student is only tentatively accepted and must be reconsidered again in the next round unless all students were accepted in the previous one.

Definite acceptances entail an opportunity cost. If a student's application is considered for some school at some round, several slots from other schools will be unavailable in the next round. This cost is not present in DA, which is consequently strategy-proof, whereas BM is not. In a deterministic context (strict priorities), the implied strategic game in BM harms efficiency, and DA performs better. However, when priorities are coarse and lotteries are most certainly needed to break ties,

[4] Data from Abdulkadiroğlu, Pathak, Roth and Sönmez (2006).

the students' intensity of preferences over schools matter,[5] and DA overlooks those.

I propose a version of BM that incorporates *Round-wise Tie-Breakers*. That is, an independent tie-breaking lottery is run each round. Accordingly, each of these rounds assigns slots in a way that can be replicated by a competitive (pseudo)market, where the price for a slot at a school equals its considered applications/remaining slots ratio. I show that this version of BM is *ex post* efficient and has several *ex ante* efficiency properties that DA does not meet.[6]

This is trivially robust to the introduction of weak priorities under the following sufficient conditions: 1) each school gives priority to a limited number of students (fewer than its capacity); and 2) each student with priority at some school prefers that school the most. These conditions are likely satisfied in the priority levels given by the presence of siblings.

While I find that DA is ordinally efficient, I also show that ordinal and *ex post* efficiency are equivalent concepts with a continuum of students. Thus, DA's ordinal efficiency does not imply superiority over BM. Indeed, DA performs very poorly if students' ordinal preferences are perfectly correlated. Precisely because DA is strategy-proof, it cannot make any distinction among students if all of them share identical ordinal preferences. In such a case, any anonymous mechanism's induced random assignment weakly *ex ante* Pareto-dominates the one given by DA.

I use simulations to study some scenarios where analytical results are hard to obtain. These new conditions tend to worsen the chances of BM against DA. They include weak priorities, correlation of students' von Neumann-Morgenstein (vNM) valuations to priorities, and naïve, truth-telling students. I find that BM outperforms DA in utilitarian *ex ante* efficiency terms even when these three problems are present at the same time. However, naïve students are better off under DA.

The concern about naïve students is relevant from a fairness perspective. Disadvantageous socioeconomic conditions are likely related to the lack of both information and time necessary to elaborate a sophisticated strategy. For naïve students, the most striking loss comes from the fact that they rank second-best choices in second position (and third-best schools in third position, and so on) when these schools might not have available slots at the corresponding assignment round. I suggest a corrective device on reported rankings at BM that would considerably alleviate this burden

[5] There are interesting attempts to empirically measure the intensity of these preferences (e.g. Black, 1999).

[6] A mechanism is *ex post* efficient if it only yields Pareto-efficient final assignments. It is *ex ante* efficient if it only yields Pareto-efficient random assignments. An ordinal mechanism is ordinally efficient if there is no other feasible ordinal random assignment that first-order stochastically dominates for all students any of the random assignments induced by the mechanism. *ex ante* efficiency implies ordinal efficiency, which implies *ex post* efficiency (Bogomolnaia and Moulin, 2001). *ex ante* efficiency is the most desirable property.

to naïve students. Corrected reported rankings would remove schools with no remaining slots to last positions. When all students are sophisticated, this correction is innocuous. Simulations show that this device works fine for naïve students while largely preserving overall efficiency.

Interest in school choice mechanisms has kept growing since the seminal paper by Abdulkadiroğlu and Sönmez (2003). The authors note that BM is not strategy-proof and suggest DA as one possible alternative. This idea has been defended since then from several fronts: theoretical (Abdulkadiroğlu, Pathak and Roth, 2005; Abdulkadiroğlu, Pathak, Roth and Sönmez, 2005; Ergin and Sönmez, 2006), experimental (Chen and Sönmez, 2006)[7] and empirical (Abdulkadiroğlu, Pathak, Roth and Sönmez, 2006). The most striking theoretical result in this line is provided by Ergin and Sönmez (2006): if schools have strict priorities, DA weakly Pareto-dominates BM.

Two recent papers are less optimistic about DA.[8] Erdil and Ergin (2008) notice that DA may lead to *ex post* efficiency losses when one relaxes the strict priority assumption. Abdulkadiroğlu, Che and Yasuda (2008) adopt a new viewpoint in this debate to which I also subscribe: introducing information about cardinal utilities into the assignment is important when priorities are not strict.

Abdulkadiroğlu *et al.* propose a new mechanism called Choice-Augmented Deferred Acceptance (CADA), which is a compromise between DA and BM. It maintains strategy-proofness with respect to revealed ordinal preferences, while it uses some information on preference intensities. This mechanism obtains the same efficiency properties BM achieves and DA does not. Since there are no major efficiency differences between BM and CADA in the scenarios I formally analyze, I discuss them in the appendix. I find scenarios in which both BM and CADA lead to the same induced random assignment. In effect, BM can be conceived as an appropriately extended CADA.

The results found here apply to several assignment problems apart from school choice. Those include residence assignment in colleges, and task/job allocation in firms and other institutions. The no-priority case shows lessons in all of these problems, despite the differences among them.

In Section 2, I briefly describe the two mechanisms I analyze in this paper. Section 3 presents basic notation and efficiency concepts. Section 4 shows the efficiency properties of DA. In Section 5, I argue that BM is *ex post* efficient and satisfies *ex ante* efficiency properties that DA does not.

[7] In constrained school choice experiments, where students must submit *limited* lists of ranked schools, Calsamiglia C., Haeringer G. and Klijn F. (2008) show that DA might not outperform BM as clearly as it does when students are allowed to submit complete rankings.

[8] Manea (2008) also questions the use of Random Serial Dictatorship (RSD), *ex ante* equivalent to DA in the scenarios I formally analyze. In many scenarios, RSD is asymptotically not ordinally efficient. He proposes the ordinally efficient Probabilistic Serial (PS) mechanism (Bogomolnaia and Moulin, 2001) instead. Further research could compare BM to PS.

In Section 6, I present simulations that test the robustness of my findings. Section 7 concludes. A discussion of CADA and the proofs are presented in the appendix.

2 The mechanisms with no priorities

2.1 Boston Mechanism, with Round-wise Tie-Breakers (BM)

In this version of BM, students report their rankings of schools. The assignment is computed in several rounds. For each possible round t, an independent fair lottery L_t assigns a real number between between 0 and 1 to each student. No two students may obtain the same number. If no corrective device is applied to protect naïve students, the assignment works as follows.[9] In the first round, the mechanism *definitely* assigns each student to her first-ranked school, in increasing order of lottery numbers according to L_1, until either school capacity is fully used or no more students rank the school as first choice. In round t, all students that were rejected at $t-1$ apply to their t-th ranked school. Then the mechanism definitely assigns these students in increasing order of lottery numbers according to L_t, again until remaining capacity is fulfilled or there are no more students to assign to that school. The process ends in a finite number of rounds, when all students are assigned.

In real life applications of BM, students obtain a unique lottery number upon application. The modification I introduce, RTB, is necessary in the following sense: with a unique lottery number, two students with identical preferences among schools with remaining slots in some round t might rank them differently. Conditional on being rejected from different schools, the student who was rejected from the least popular one must have a bad lottery number. Hence she faces lower assignment probabilities (higher "prices") than the other student. By running an independent lottery each round, I make sure that students are facing the same "prices" each round. I show that *ex post* efficiency is ensured in this modified BM.

2.2 Deferred Acceptance (DA)

In DA, students report their rankings of schools. An even lottery assigns a real number between 0 and 1 to each student. Again, each student obtains a different number. The assignment is computed in several rounds. In the first one, the mechanism *tentatively* assigns each student to her first-ranked school, in increasing order of lottery numbers, until either school capacity is reached or no

[9] The theoretical results of this paper are based on the assumption that all students are sophisticated. The corrective device will not be analyzed until Section 6.

more students rank the school as first choice. In each remaining round, each student applies to her (reportedly) most-preferred school among the ones that have not rejected her. The mechanism compares *all* students who are applying to the same school, and tentatively reassigns students in increasing order of lottery numbers, again until the capacity limit is reached, or no more slots are demanded. The process ends when all students are assigned.

The fact that acceptances are tentative is what makes DA strategy-proof. In BM, suppose that a student moderately prefers school 1 to school 2, and yet knows that school 1 is much more demanded than school 2. She may put school 2 ahead in her reported ranking. Since slots are definitely assigned each round with BM, applying to one school has an oppotunity cost at other schools. This does not happen with DA, because all slots are again available in each round.

3 Notation and concepts

There is a finite set S of J schools $S = \{1, ..., J\}$. For simplicity I assume that there is no outside option.[10] Each school j has capacity measure $\eta_j > 0$. The total sum of capacities across schools is 1. Let $\vec{\eta} = (\eta_1, ..., \eta_J)$. There is a measure 1 of student types. Each type is associated with a von Neumann-Morgenstein (vNM) utility vector $v = (v_1, ..., v_J) \in V \equiv \Delta^{J-1}$.[11] The set of student types, V, is equipped with a (full support) probability measure over sets of student types, denoted as $m : B(V) \to [0, 1]$, where $B(V)$ is the Borel-algebra associated with V. I assume that $m(\{v\}) = 0 \; \forall v \in V$ though this is not necessary for some of the results. The pair $(m, \vec{\eta})$ defines the economy, and E denotes the set of all economies satisfying the conditions above. I use the function $o : \{v \in V : \nexists i, j \in S, v_i \neq v_j\} \to \Pi(S)$ to indicate the ordinal preferences associated with students' types, where $\Pi(S)$ is the set of all permutations over S. $o_j(v)$ is the j-th highest element in v.

A set of fair lotteries assigns to each student her own element $e \equiv (e_1, ..., e_N)$ in the lottery set $L = [0, 1]^N$, where N is the number of lotteries the mechanism needs to break all possible ties. No two students hold the same lottery element. For the purposes of the present analysis, it is enough with $N = J$.[12] All components of e are independent from each other. The set L is accompanied by a uniform measure l such that, for any measurable subset $\bar{L} \subset L$, $l(\bar{L})$ is the volume of $\bar{L}$. For any such $\bar{L}$, the conditional probability measure of student types among students holding their lottery

[10] The results here presented could easily be extended to include that option.

[11] Rescaling the type space into the J-dimensional simplex is innocuous in that it does not alter equilibrium strategies.

[12] That is the maximum number of rounds BM needs to assign all students. Not all lottery element components are necessarily used in the assignment. For instance, DA only uses e_1.

elements in $\bar{L}$ is equivalent to m. One can think of a student as a pair $(v, e) \in V \times L$.

An *assignment* is a measurable function $a : V \times L \to S$. An assignment is feasible at $(m, l, \vec{\eta})$ if for any school j, the (product) measure $m \times l$ of students who are assigned to that school equals η_j. A *random assignment* is a measurable function $q : V \to \Delta^{J-1}$. A random assignment q is feasible at $(m, \vec{\eta})$ if $\int_V q(v)m(dv) = \vec{\eta}$. By the Birkhoff - von Neumann theorem, any random assignment can be implemented as a lottery over assignments. Each v-type student's expected payoff from the random assignment $q(\cdot)$ is equal to $q(v) \cdot v$, where $q_j(v)$ is the probability that a v-type student is assigned to school j. A random assignment is *ordinal* if for any $v, \tilde{v} \in V$ such that $o(v) = o(\tilde{v})$, we have $q(v) = q(\tilde{v})$. A random assignment is *cardinal* if it is not ordinal.

A feasible random assignment is *ex post efficient* at $(m, \vec{\eta})$ if any assignment from it is Pareto-optimal among all feasible assignments at $(m, l, \vec{\eta})$. That is, for any possible lottery outcome, the resulting assignment is Pareto-optimal. A feasible random assignment is *ex ante efficient* at $(m, \vec{\eta})$ if there is no other feasible random assignment at $(m, \vec{\eta})$ that provides each student type with weakly higher expected payoff and a positive-measure set of types obtains strictly higher payoff.[13] A feasible ordinal random assignment $q(\cdot)$ is *ordinally efficient* at $(m, \vec{\eta})$ if there is no other feasible ordinal random assignment $\tilde{q}(\cdot)$ that first-order stochastically dominates $q(\cdot)$ for any student type v.[14]

For ordinal random assignments, *ex ante* efficiency implies ordinal efficiency, which implies *ex post* efficiency. The converse is not always true (Bogomolnaia and Moulin, 2001). For cardinal random mechanisms, *ex ante* efficiency implies *ex post* efficiency, and the converse is not always true.

A final notion of efficiency is based on a particular subset of schools. A random assignment $q(\cdot)$ is *ex ante efficient within a set* $S' \subset S$ of schools if, for any other feasible random assignment $\tilde{q}(\cdot)$ that keeps assignment probabilities unchanged with respect to the schools in $S \backslash S'$, $\tilde{q}(\cdot)$ does not provide each student type with weakly (and strictly in a positive-measured set of types) higher expected payoff than $q(\cdot)$.

For a fixed economy $(m, \vec{\eta}) \in E$, an (anonymous) *mechanism* M consists of a finite strategy space Σ for each student type and a functional $\tilde{M}_M : \Sigma^V \times L \to A$, where A is the set of all measurable functions $\alpha : \Sigma \times L \to S$. Each function of this set assigns students to schools depending on their

[13] The literature on mechanism design refers to this as *interim efficiency*, since agents have already learned their types at this point. I follow the previous school choice literature in this definition of ex ante efficiency (for instance, Abdulkadiroğlu, Che and Yasuda, 2008).

[14] First-order stochastic dominance arises when $\sum_{i=1}^{j} \tilde{q}_{o_i(v)}(v) \geq \sum_{i=1}^{j} q_{o_i(v)}(v)$ for any $j \in \{1, .., J\}$, and $\tilde{q}(v) \neq q(v)$.

strategies and their lottery elements. Any mechanism M has an associated *random mechanism* $\mu_M : \Sigma^V \to R$, where R is the set of all measurable functions $\rho : \Sigma \to \Delta^{J-1}$, such that $\mu_M = \mathbf{E}_l(\tilde{M}_M)$. A (pure) strategy profile given M is a measurable function $\sigma : V \to \Sigma$. A random mechanism μ_M is *feasible* at $(m, \vec{\eta})$ if for any pure strategy profile σ there is $\rho \in R$ such that $\mu_M(\sigma) = \rho$ and $\int_V \rho(\sigma(v))m(dv) = \vec{\eta}$. A mechanism is feasible if its associated random mechanism is feasible.

A feasible mechanism M *induces (in pure strategies)* a random assignment q at $(m, \vec{\eta})$ if given that economy and that mechanism there is $\rho \in R$ and a pure strategy Nash equilibrium profile σ^* such that $\mu_M(\sigma^*) = \rho$ and $\rho(\sigma^*(v)) = q(v)$ for any $v \in V$. A mechanism M is efficient at $(m, \vec{\eta})$ in any of the senses depicted above if any random assignment $q(\cdot)$ induced by the mechanism at $(m, \vec{\eta})$ is efficient in the same sense. A mechanism is ordinal at $(m, \vec{\eta})$ if it induces ordinal random assignments only.

4 DA is ordinally efficient, yet...

DA with a continuum of students is shown to be strategy-proof by Abdulkadiroğlu, Che and Yasuda (2008).[15] While DA is *ex post* Pareto-dominant among stable mechanisms if schools have strict priorities (Gale and Shapley, 1962), DA may yield *ex post* efficiency losses when schools have weak priorities (Erdil and Ergin, 2008). Erdil and Ergin (2008) provide the mechanism designer with an improved mechanism that guarantees *ex post* efficiency among stable assignments. The drawback is that *ex post* efficient-stable assignments are Nash-implementable only, so the improved mechanism is not necessarily strategy-proof.

My first result indicates that DA is ordinally efficient if there is a continuum of students and schools have no priorities. This is shown by means of proving its equivalence to the Probabilistic Serial (PS) mechanism. The latter is defined as a random assignment mechanism, via the "cake-eating" algorithm with equal speeds. Students are requested to report their rankings over schools. At each moment of time, each student "eats" shares of a slot at her reportedly most-preferred *available* school at speed 1. A school becomes unavailable when the measure of eaten slots equals the school capacity. At time 1, all schools have become unavailable, and the process ends. The assignment probability of a student to a school is the total share of a slot at that school the student has eaten.

PS is ordinally efficient, as shown by Bogomolnaia and Moulin (2001), but it is not always strategy-proof. Kojima and Manea (2007) have shown that PS is strategy-proof if schools' capacities

[15] Dubins and Freedman (1981) and Roth (1982) show it for a finite number of students.

are high enough.

Proposition 1 *With a continuum of student types and no school priorities, DA and PS are ex ante equivalent.*

Proof. See appendix. ■

Corollary 1 *With a continuum of agents and no school priorities, DA is ordinally efficient.*

This result has also been found by Che and Kojima (2008),[16] for the asymptotic case where the number of students (and also of school slots) grows unboundedly large. The proof of Proposition 1 is fairly simple. With a continuum of students, Abdulkadiroğlu, Che and Yasuda (2008) have shown that any assignment induced by DA can be characterized by a vector of cutoffs, one for each school. A student is assigned to a school she applies to (after having been rejected at other preferred schools) if and only if her assigned lottery number is lower than the school's cutoff. Likewise, one could characterize the random assignment induced by PS as a vector of "cake-end" times,[17] one for each school. Each school becomes unavailable at its "cake-end" time. The proof concludes by observing that both cutoffs and "cake-end" times are the same.

For the defenders of purely ordinal mechanisms, this result is very positive. Ordinal efficiency is the most one can demand from a mechanism that relies on (truly reported) ordinal preferences only. However, I provide a second result that neutralizes the importance of ordinal efficiency relative to *ex post* efficiency. This is related to Manea (2008), which states that Random Serial Dictatorship (*ex ante* equivalent to DA in the present context) is ordinally efficient if and only if *ex post* efficiency implies ordinal efficiency.

Theorem 1 *With a continuum of students and no priorities, an ordinal mechanism is ordinally efficient if and only if it is ex post efficient.*

Proof. See the appendix. ■

BM is not an ordinal mechanism, since two student types with identical ordinal preferences may play different strategies in equilibrium. Thus, BM cannot be compared to DA in ordinal efficiency terms. In the light of Theorem 1, however, a comparison can be made. Ordinal efficiency does not claim any superiority for DA over BM, if BM is *ex post* efficient.

[16] See Kesten (2007) for a related result in housing allocation problems.

[17] Che and Kojima (2008) use the term "expiration dates".

Abdulkadiroğlu, Che and Yasuda (2008) show that, under the full support assumption, DA cannot be *ex ante* efficient within any subset of more than two schools.[18] Besides, DA does not make any use of the intensity of students' preferences. The following result strongly illustrates to which extent this may lead to efficiency losses. Let us say that *a mechanism M is abysmal* in an environment $(m, \vec{\eta})$ if for any other (anonymous) mechanism, all of its induced random assignments at $(m, \vec{\eta})$ weakly *ex ante* Pareto-dominate any random assignment induced by M at $(m, \vec{\eta})$.

Proposition 2 *With a continuum of students with identical ordinal preferences, and no priorities, DA is an abysmal mechanism.*

Proof. See the appendix. ■

This interesting result is an extension of Theorem 6 in Abdulkadiroğlu, Che and Yasuda (2008), which states that CADA *ex ante* Pareto-dominates DA in those scenarios.

Proposition 2 goes further than the aforementioned authors' result by showing that, when information on ordinal preferences is not useful, any (anonymous) cardinal mechanism, *no matter how badly designed*, leads to (weak) *ex ante* payoff improvements for all students, as compared to ordinal mechanims. The underlying assumption (identical ordinal preferences) is not that far from real scenarios. The concern about observable quality differences across schools is precisely one of the motivations for school choice programs.[19]

5 BM obtains good *ex ante* efficiency results

I next show that the Boston Mechanism satisfies several *ex ante* efficiency properties that DA may not meet. Given m, $\vec{\eta}$ and an (anonymous) mechanism M, a school j is *overdemanded in an equilibrium* σ^* if, denoting $\rho^* = \mu_M(\sigma^*)$, there is no pure strategy $\varsigma \in \Sigma$ such that $\rho^*_j(\varsigma) = 1$. A school j is underdemanded if it is not overdemanded. In words, a school is overdemanded in equilibrium if there is no means by which a student could obtain sure assignment at that school, given the other students' equilibrium strategies.

Let a (pure) ranking strategy for a v-type in BM be a vector $r(v) \in \Pi(S)$.[20] Let $r_j(v)$ be

[18] The authors show that there is a positive-measured set of student types who obtain positive assignment probability for any school, under DA. Consider any set of three schools. These students could always trade assignment probabilities for these schools among them in a mutually profitable way.

[19] For instance, under the No Child Left Behind regulation, school districts are obliged to provide parents with alternative choices, if the school their children are attending to do not meet minimum quality standards.

[20] Recall that $\Pi(S)$ is the set of all possible permutations on $\{1, ..., J\}$.

the school that a v-type student ranks in j-th position. A pure strategy BM Nash equilibrium $r^* : V \to \Pi(S)$ exists (see Theorem 2 in Mas-Colell, 1984). It follows from the definition above that a school j is *overdemanded in a BM equilibrium* r^* if $m(\{v \in V : r_1^*(v) = j\}) > \eta_j$.

Proposition 3 *For any fixed* m *and* $\vec{\eta}$*, BM satisfies the following properties:*

1) It is ex post efficient.

2) It is ex ante efficient within the set of overdemanded schools.

3) It is ex ante efficient if there is only one underdemanded school.

4) It is ex ante efficient if $J = 3$*, all students have the same ordinal preferences* $o(v) = (1, 2, 3)$ $\forall v$*, and* $m\left(\left\{v \in V : v_2 > \sum_j v_j \eta_j\right\}\right) \geq \eta_2$.

Proof. For conditions 1, 2 and 3, see the appendix. Condition 4 follows from Lemma 1 (also in the appendix). ■

The reader will observe that CADA meets all these conditions as well, as shown by Abdulkadiroğlu, Che and Yasuda (2008). The appendix provides a wider comparison between BM and CADA.

The idea of condition 2 is that there is a "market logic" in r_1^* for those who choose overdemanded schools as first options. For each overdemanded school there is an "equilibrium price" which equals the demand/supply ratio. The proof follows from constructing the proper environment in which Hylland and Zeckhauser's (1979) (symmetric) pseudomarket mechanism (PM),[21] which is *ex ante* efficient, reaches the same random assignment as BM does, with respect to overdemanded schools.

ex post efficiency arises partly because of Round-wise Tie-Breakers. RTB separates rounds so that individual assignment probabilities in further rounds are not affected in any relevant way by individual choices in previous rounds. Each separate round follows the same "market logic" with the remaining slots as the first round does with all of them. We obtain *ex ante* efficiency within each set of schools that become full at the same round. Additionally, no profitable exchanges can come across rounds: if a student prefers an underdemanded good to an overdemanded one, she would never rank the overdemanded school first. Thus, she would not have probability shares of overdemanded schools. All this results in acyclicity: *ex post*, it is not possible to design a Pareto-improving trading cycle.

[21] In the symmetric PM, each student is given a budget of one "fake" money unit, which she spends in buying assignment probabilities. There is at least one pseudomarket price equilibrium. It follows that a school is overdemanded in PM equilibrium if its corresponding price is higher than 1.

This argument does not imply that BM is *ex ante* efficiency. A student may optimally choose an underdemanded school, but be willing to trade probability shares of it with students who apply to other underdemanded schools in further rounds. Relative prices vary across rounds. For instance, two underdemanded schools may become full in the very next round, with different demand-to-remaining-capacity ratios. This variation of relative prices across rounds explains the lack of *ex ante* efficiency.

Condition 3 is proven through the following remark, which is a corollary from the proof for condition 2. Let $Q^M(m, \vec{\eta})$ denote the set of (pure strategy) equilibrium random assignments that mechanism M induces given the measure m and the capacities $\vec{\eta}$. Let $Q^M(m, \vec{\eta}; s)$ denote the set of equilibrium random assignments that mechanism M induces given the measure m and the capacities $\vec{\eta}$ where there are exactly s underdemanded schools in equilibrium.

Remark 1 *Fix m and $\vec{\eta}$. Then $Q^{BM}(m, \vec{\eta}; 1) = Q^{PM}(m, \vec{\eta}; 1)$.*[22]

A final note on Proposition 3 makes special reference to popular schools. A school j is *popular* if $m\left(\{v \in V : v_j = \max_{i \in S} v_i\}\right) > \eta_j$.[23] Both in BM and in CADA, any equilibrium is characterized by the fact that all the popular schools are overdemanded. Hence, both BM and CADA are always *ex ante* efficient within the set of popular schools. If there are more than two popular schools, DA cannot possibly be *ex ante* efficient within that set.

The reader may wonder if Proposition 3 holds after the incorporation of weak priorities. Admittedly, priorities alter incentives in such a way that BM's "nondiscriminatory market" logic fails. Simulations presented in next section cope with that problem, since a formal analysis is not feasible. I nevertheless comment on a kind of priority level that resembles *sibling priority.*

Remark 2 *In a scenario with weak priorities, suppose that: 1) for each school, the measure of students with priority there is lower than the capacity the school offers; 2) there is a perfect match between each student a school gives priority to and the school the student's parents most prefer. Then, BM still meets properties 1 through 3 in Proposition 3.*

Both assumptions are plausible in the sibling priority case.[24] Under these assumptions, each student with priority status is surely placed at her most-preferred school and removed from the

[22] See the proof of Lemma 1 (in the appendix).

[23] Notice that this concept is not linked to any specific equilibrium or mechanism.

[24] Again citing data from Abdulkadiroğlu, Pathak, Roth and Sönmez (2006): only 21% of assigned students in Boston elementary schools had either sibling or sibling-walk priority, as of 2001-02.

mechanism. The real assignment problem involves the remaining students without priority. It follows that all the results presented here are robust to the inclusion of this kind of priorities.

6 Simulations

Although some qualitative efficiency properties have been compared and discussed across the three mechanisms, they tell us little about the quantitative *ex ante* efficiency gains that BM can achieve relative to DA, if any. Additionally, a natural question concerns whether my results are robust to the inclusion of weak priorities, beyond the conditions in Remark 2. Finally, it is interesting to consider scenarios where a portion of students' parents are naïve truth-tellers.[25]

All these questions cannot be directly addressed via analytical tools. Therefore, I have designed simulations that compute (utilitarian) welfare measures for all three mechanisms.[26] I have considered different scenarios, varying the number of schools, the capacity of each (all schools with identical capacity), the correlation of vNM valuations among students, whether there are naïve students or not, whether schools have priorities or not, and whether valuations are correlated with school priorities or not.

The first subsection gives results when no priorities or naïve agents are considered. The second subsection analyzes scenarios with naïve agents. The third includes priorities, with and without naïve students. Additional simulation results comparing BM to CADA are shown in the appendix.

6.1 No priorities, no naïve students

I consider scenarios with 4, 5 and 6 schools, and 20, 30 and 40 slots per school.[27] The total number of students equals the total number of slots. Each student i is independently endowed with a private vNM valuation vector v_p^i that is drawn from the J-dimensional uniform distribution. There is a common value vector v_c drawn from the same distribution. Student i's final valuation vector is

[25] Naïveté might be understood as the result of a scarcity of time and information, rather than a lack of ability. Wealthier parents might have more time to gather useful information and to design a strategic school choice application. This raises questions about fairness (Abdulkadiroğlu, Pathak, Roth and Sönmez, 2006; Pathak and Sönmez, 2008).

[26] All codes can be found at http://people.bu.edu/miralles.

[27] In Abdulkadiroğlu, Pathak, Roth and Sönmez (2006) data, there were 40 incoming students per Boston elementary school, as of 2001-2002. In Cambridge, MA, as of the 2008-09 kindergarten assignment, the average capacity was around 33 vacant slots per institution. Abdulkadiroğlu, Che and Yasuda (2008) focus on the 5 school - 20 students/school case. Some of my figures follow their lead so as to make results comparable.

$v_a^i = av_c + (1-a)v_p^i$, where $a \in \{.1, .3, .5, .7\}$. a proxies the correlation among all students' valuation vectors. After having obtained all valuation vectors, I rescale them to the simplex.

I compute pure strategy Nash equilibria of BM (also of CADA) through iterated best responses, where truth-telling is the starting point. I acknowledge the possible existence of multiple Nash equilibria.[28] To avoid limit cycles, I establish rigidities by which randomly chosen students may keep their strategies unchanged between iterations even if they wish to switch to a new strategy. If no Nash equilibrium is found after 2,000 iterations, convergence is called when the strategy matrices (each one with dimension equal to the number of students times the number of schools) between two consecutive iterations have at least 98% coincidences. With five schools, this percentage means that at most 5% students are willing to change their strategies. In computing BM random assignments given the students' chosen strategies, I take a continuum approach that approximates assignment probabilities by considering a student as if she were an atom of students. This method is less time-consuming than the one of averaging assignments over a big sample of lottery draws. Analogous time savings occur when DA random assignments are calculated via the cutoffs characterized by Abdulkadiroğlu, Che and Yasuda (2008).

After having computed Nash equilibria and their corresponding random assignments, I calculate the mean of *ex ante* payoffs across all students, which accounts as my measure for utilitarian *ex ante* efficiency. I proceed likewise for 100 independent draws of all students' vNM valuations (plus common value component), and I average the welfare measures over that sample of 100 draws, in order to obtain a final welfare estimate.[29]

Figures 1 and 2 illustrate the differences in computed welfare between BM and DA. In both figures, it is apparent that the relative welfare gain stemming from a switch from DA to BM is always positive and increasing in the correlation among students' valuation vectors.[30] The greater this correlation, the more similar ordinal preferences become across all students, and the worse DA performs. Figure 1 suggests that welfare gains positively depend on the number of schools, as more schools imply broader strategic possibilities.[31] Figure 2 shows that these gains are not very sensitive to a variation in the capacity of each school.

[28] This problem may not be present in plausible scenarios (Pathak and Sönmez, 2008).

[29] After 4000 iterations, the process had not converged in a 2% of cases in BM, and a 2.92% in CADA. Respective average convergence ratios, conditional on not convergence, are 93.01% and 92.99%.

[30] Gains follow the same trend and similar rates as the ones reported in a previous version of Abdulkadiroğlu, Che and Yasuda (2008), where they compare CADA to DA.

[31] A preliminary 75-draw simulation with seven schools tends to confirm this trend: BM obtains a 4.94% welfare increase with respect to DA, in a 20 students/school scenario when $a = .7$.

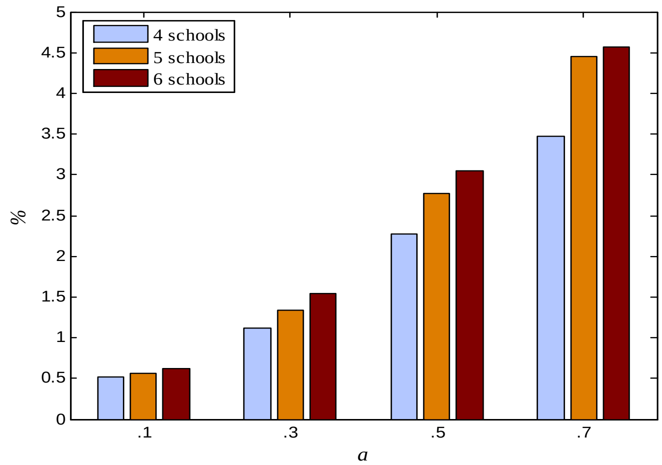

Fig. 1: % Welfare gain BM-DA, 20 students/school.

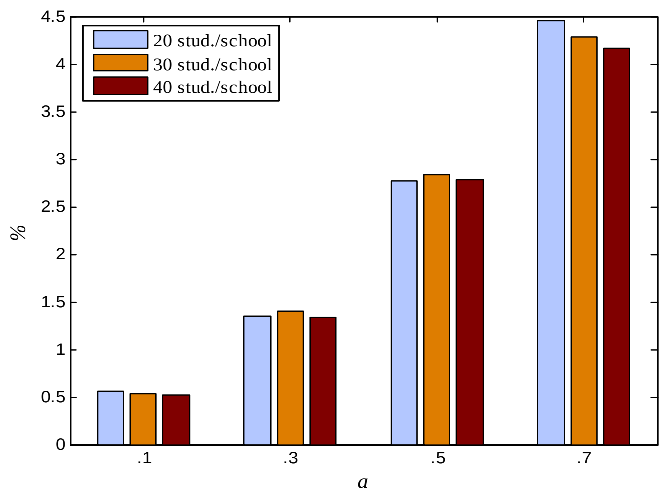

Fig. 2: % Welfare gain BM-DA, 5 schools.

6.2 No priorities, naïve students

I calculate equilibria[32] and welfare again, under the assumption that half the students are naïve truth-tellers.[33] Additionally, I have computed welfare when the following corrective device is applied

[32] After 2,000 unsuccessful iterations, convergence is called after observing 99% coincidences between consecutive iterations. After 4000 iterations, the process had not converged in a 1.14% of cases in BM, and a 2.28% in CADA. Respective average convergence ratios, conditional on not convergence, are 97.52% and 96.45%.

[33] According to Abdulkadiroğlu, Pathak, Roth and Sönmez (2006), 35.7% parents ranked two overdemanded Boston elementary schools in the first two positions, in the 2001-02 assignment. This is a naïve strategy, since the second

to protect naïve students. Whenever a student applies to a school that has no remaining slots, that school is removed to the last position of her ranking, and all schools ranked below gain one position in that ranking. Hence, the student applies to the next-ranked school.[34]

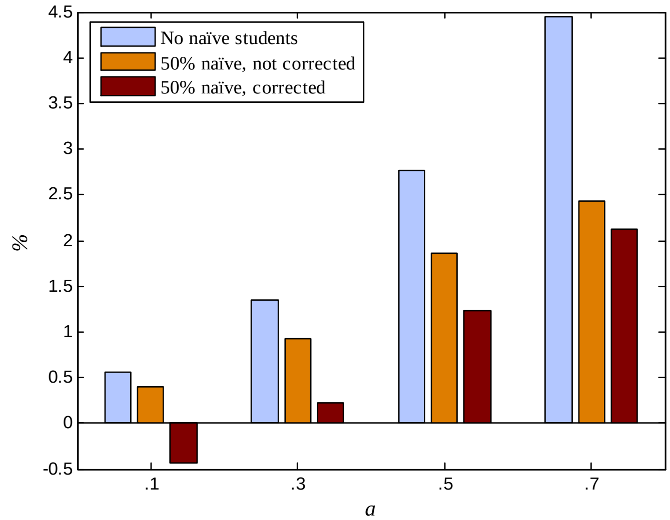

Fig. 3: % Welfare gains BM-DA, 5 schools, 20 students/school.

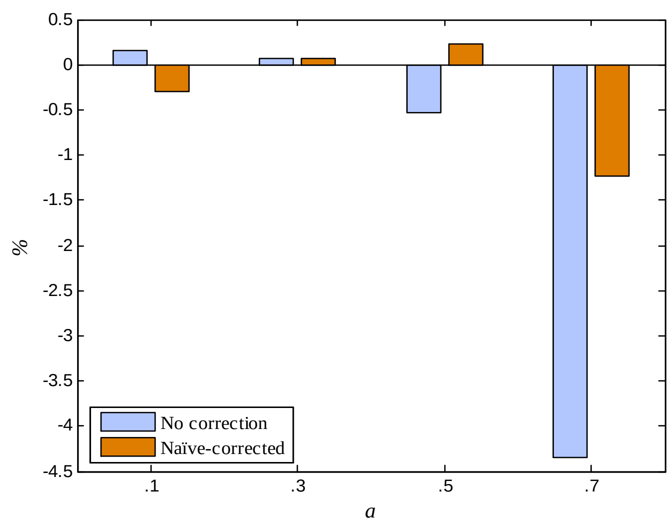

Fig. 4: Naïve students' % welfare differences BM-DA, 5 schools, 20 stud./sch.

choice is wasted. A substantially but not overwhelmingly higher amount of naïve students represents a bad scenario for BM. If all students were naïve, the outcome of BM would be ex post efficient.

[34]For the corrected-BM case, the sample is constituted by 75 valuation draws.

Figure 3 shows that when half the students are naïve truth-tellers, BM's relative welfare gains with respect to DA are cut approximately by half. Still, utilitarian efficiency gains subsist and are increasing in the correlation among students' valuations. The protective device for naïve students has an additional efficiency cost. However, it becomes less relevant as the correlation among students' valuations increases.

Figure 4 shows that naïve students are punished more in BM than in DA, quite significantly when the correlation among students' vNM vectors is high. The idea is that ordinal preferences tend to coincide among students, and naïve students rank their actual second-best school in second position, their actual third-best school in third, and so on. Some of these options are likely to be overdemanded, in which case naïve students are wasting their choices. This does not happen in DA, since acceptances in each round are tentative. The corrective device shows its effectiveness: with $a = .7$, naïve students' welfare losses are reduced from near 4.5% to less than 1.25%.

That BM is more utilitarian-efficient than DA comes from the sophisticated players' performance, who are facing fewer strategic opponents. They do better in BM than DA to an extent that exceeds the damage naïve students suffer. In that sense, figures 3 and 4 are in line with Pathak and Sönmez (2008). With a substantial number of naïve students, sophisticated students are benefited by BM, and naïve students would suffer on average a payoff loss under BM, relative to DA.

6.3 Weak priorities

I introduce one level of school priorities in the simulation model, to see if the results are robust to this. Priorities in one school are independent from priorities in the others. For a given school, a student's priority status is independent from other students' status. The priority status is drawn from a Bernoulli distribution with a success probability that equals the inverse of the number of schools. Since schools' priorities are mutually independent, it might be the case that the same student has priority at several schools. The expected number of students with priority at some given school equals its capacity. This is higher than what one could expect from sibling priority, which was still a fine priority structure for BM (see Remark 2).[35]

In a first set of scenarios considered, students' vNM valuations and priority status are uncorrelated. In an alternative set, student i's vNM valuation for a school j where the student has priority is increased to $.2 + .8v^i_{aj}$, while her valuation at a school h that does not give priority to her is reduced

[35] A student's probability of not having priority at any school is similar among the number of schools here considered (.3349 with 6 schools, .3277 with 5, .3164 with 4).

to $.8v^i_{ah}$. In this way, I incorporate the realistic assumption that school priority criteria (walking distance, siblings,...) are correlated with students' vNM valuations. Valuations are afterwards rescaled to the simplex. Possible values of a are reduced to the set $\{.2, .6\}$.

Nash equilibria of BM are calculated in each scenario for 75 draws of all students' private valuations plus common vNM vector.[36] Given students' revealed rankings, assignment probabilities can be approximately calculated using the continuum approach of previous subsections. For each of these draws and scenarios, the DA random assignment is also computed. Nevertheless, the continuum approach cannot be used in DA, since analytical difficulties arise when students apply first for schools at where they do not have priority. Since no equilibrium calculation is needed in DA, allocation probabilities are still computable in short time by drawing 500 lottery outcomes, calculating the assignment for each lottery, and averaging over them. The next figures show several welfare comparisons, where welfare is computed as the average of mean (equilibrium) payoffs across 75 valuation draws.

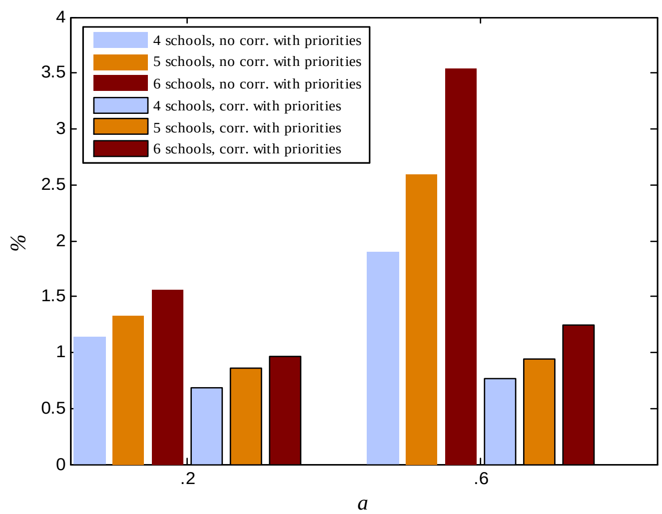

Fig. 5: % Welfare gain BM-DA, weak priorities, 20 stud./school.

[36] Convergence criteria are identical to the ones applied with no priorities. After 4000 iterations, the process had not converged in a .67% of cases with no naïve students, and a .85% with naïve students. Respective average convergence ratios, conditional on not convergence, are 92.68% and 97.98%.

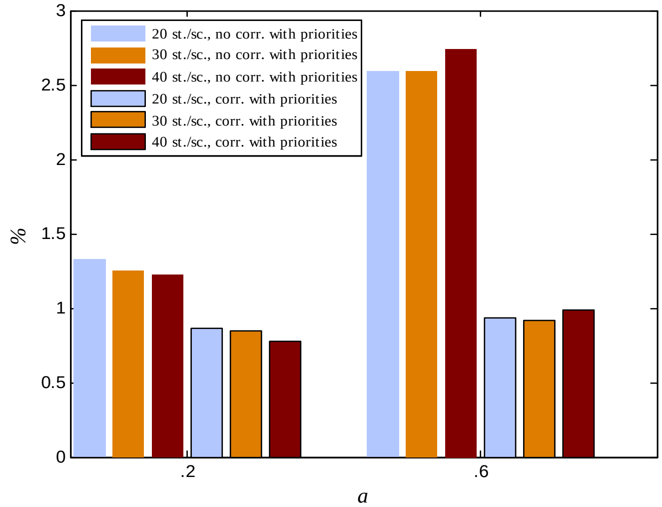

Fig. 6: % Welfare gains BM-DA, weak priorities, 5 schools.

Figures 5 and 6 compare BM to DA in scenarios with no naïve students. BM is more utilitarian-efficient in all cases. Welfare gains are higher if valuations are not correlated to school priorities. In this latter case, they are also more sensitive to an increase in the correlation among students' valuations. Interestingly, they remain quite unaffected by the introduction of weak priorities alone, as a rapid observation of figures 1 and 2 reveals.

That the correlation of valuations to priorities reduces the sensitivity of welfare gains to the correlation of valuations among students is explained by the fact that priorities are independent across students. This independence induces a reduction of correlation among *all* students' valuations. It instead tends to segment students with respect to most-preferred schools, evenly across schools. This lowers competition in any of the mechanisms, so welfare differences diminish in a correlation-to-priorities scenario because DA performs sufficiently well.[37]

Next, I introduce naïve students in this scenario with weak priorities. BM still outperforms DA with respect to utilitarian welfare, with or without the corrective device that protects naïve students, as can be seen in figure 7. Interestingly, welfare gains are enhanced by the corrective device. No trade-off arises between fairness improvement and efficiency gains, in constrast to what happens in a no-priority scenario. In other words, sophisticated students' welfare gains suffer little harm when

[37] An example with 5 schools and 20 students per school. With $a = .2$, DA-generated welfare without correlation to priorities is .2945, which slightly grows to .2972 with correlation. With $a = .6$, the improvement is clearer: from .2237 without correlation to .2494 with it.

naïve students are protected.

This may be explained by the fact that naïve students are less protected by the corrective device in the presence of weak priorities (figure 8). The corrective device is able to reduce the naïve students' welfare loss. However, some of this welfare loss still remains because the corrective device does not prevent naïve students from applying to schools where students without priority status have no chance.

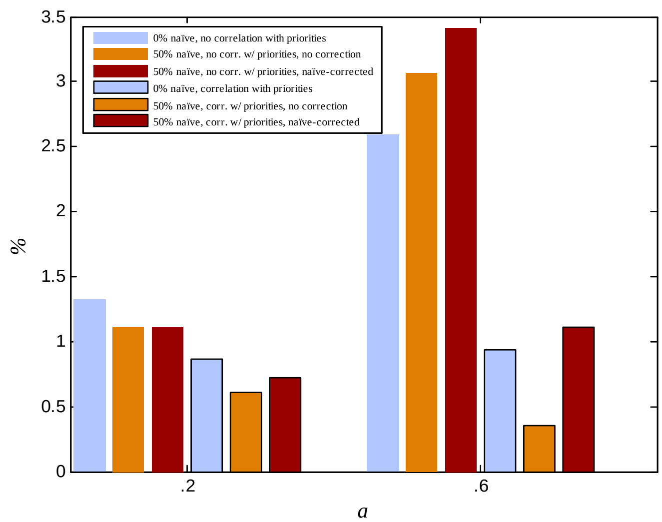

Fig. 7: % Welfare gain BM-DA, weak priorities, 5 schools, 20 stud./sch.

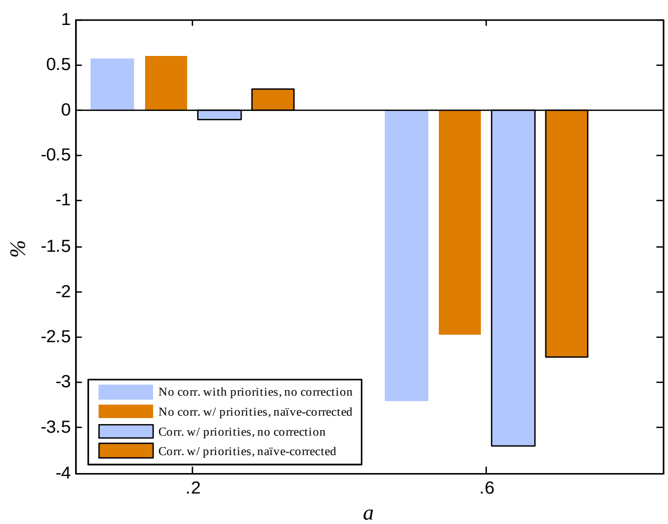

Fig. 8: Naïve students' % welfare gain BM-DA, weak prio., 5 sch., 20 st./sch.

7 Conclusion

A large literature advocates replacing the Boston Mechanism in School Choice programs. The Deferred Acceptance algorithm has been implemented in its place in Boston and Seattle. I introduce several elements to this debate that might support the persistence of BM in other municipalities such as Minneapolis or Cambridge, MA.

I analyze scenarios where schools have no priorities over students. Some strong theoretical results against BM are based on the assumption that schools have strict priorities over all students. In real-life school choice, students are classified according to a maximum of four-to-five priority levels. Thus my approach seems less far from reality than the assumption of strict priorities. In addition, it allows for the analysis of realistic situations where students are potentially tied in terms of entry priority.

With a continuum of students, DA is ordinally efficient. However, I show under the same conditions that this property is not superior to *ex post* efficiency. Moreover, DA performs very poorly when all students share the same ordinal preferences over schools. Any other anonymous mechanism weakly *ex ante* Pareto-dominates DA in that case.

Under full support of vNM utilities, DA is not *ex ante* efficient if the number of schools exceeds two (Abdulkadiroğlu, Che and Yasuda, 2008). I propose a BM with Round-wise Tie-Breakers (RTB) that is *ex post* efficient and achieves several improvements over DA regarding *ex ante* efficiency.

Simulations support the idea that BM is more (utilitarian) *ex ante* efficient than DA. The superiority of BM over DA in the utilitarian sense is maintained even when weak priorities, correlation of valuations to priorities, and naïve students are included.

Naïve students are however better off under DA. If BM is to be kept or adopted, the concern about naïve students has considerable relevance. For naïve students, a substantial loss comes from the fact that they rank second-best choices in second position (and third-best schools in third position, and so on) when these schools might not have available slots at the corresponding assignment round. A partial solution might be to introduce a corrective device on reported rankings. Corrected reported rankings would remove schools with no remaining slots to last positions. This correction would not alter Nash equilibria outcomes when all students are sophisticated. Simulations suggest that this device effectively protects naïve students.

It is common practice in BM to assign a unique lottery number to each applicant. Should BM be maintained or implemented, I recommend that each assignment round be accompanied by an independent tie-breaking lottery. With no priorities, RTB makes each assignment round work as a nondiscriminatory market, and this fact preserves *ex post* efficiency, in a no-priorities context.

Abdulkadiroğlu, Che and Yasuda (2008) have proposed a new Choice-Augmented DA mechanism as a compromise between BM and DA. Municipal authorities may need to proceed cautiously when choosing among these mechanisms (DA, CADA and BM). If school priorities are strict in most cases, then DA appears to be the best mechanism. At the other extreme, when schools have a low number of priority levels, BM could be chosen or CADA could be a good alternative.[38]

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8 Appendix

8.1 Comments on CADA (Choice-Augmented DA)

Abdulkadiroğlu, Che and Yasuda (2008) propose this new assignment mechanism in school choice in an aim to find a compromise between DA and BM. In a setup with no priorities, CADA works as follows. First, each student reports her ranking of schools *plus a target* school. Two independent lotteries L_T and L_R are run. School priorities over students are constructed via lexicographic tie-breakers: first, the fact that the student has targeted the school; second, among students who target the same school (and if they simultaneously point to this one), the lottery L_T; finally, for other ties, L_R. The process continues with a DA algorithm in which school priorities determine the order of the students' tentative assignments.

CADA is strategy-proof with respect to reported rankings (any DA procedure is). The only strategic element here consists of the school the v-type student targets. A targeting strategy for this type is denoted as $\tau(v)$. Unlike the original paper, I focus on pure strategies, since a Nash equilibrium in pure strategies $\tau^* : V \to S$ exists in the induced targeting game (Abdulkadiroğlu, Che and Yasuda, 2008). Define a school j as *overdemanded in a CADA equilibrium* τ^* if $m(\{v \in V : \tau^*(v) = j\}) > \eta_j$. A school j is underdemanded if it is not overdemanded. As mentioned in Section 5, CADA satisfies all the properties enumerated in Proposition 3.

The following Lemma directly proves condition 4 in Proposition 3. Recall previously used notation. $Q^M(m, \vec{\eta}; s)$ denotes the set of equilibrium random assignments that mechanism M induces given the measure m and the capacities $\vec{\eta}$ where there are exactly s underdemanded schools in equilibrium.

Lemma 1 *Fix m and $\vec{\eta}$. Then $Q^{BM}(m, \vec{\eta}; s) = Q^{CADA}(m, \vec{\eta}; s)$, for $s = 1, 2$.*

Proof. See the second part of the appendix. ■

An important point here is that *CADA and BM provide exactly the same outcomes when the number of underdemanded schools is sufficiently low.* Differences arise between CADA and BM only when the number of underdemanded schools is higher than two. With a total of three schools, it follows that BM and CADA perform identically. Since condition 4 in Proposition 3 is shown for CADA by Abdulkadiroğlu, Che and Yasuda (2008), it obviously holds for BM.

When both BM and CADA are *ex ante* efficient due to the fact that there is only one underdemanded school, it turns out that each resulting random assignment is indeed one arising from the pseudomarket mechanism.

Since BM performs qualitatively similarly to CADA, the major justification for this new mechanism must come from its simplicity. Clearly, it is easier to choose a single school to target than to choose a complete ranking. This is a significant advantage for CADA: *ex ante* efficiency gains are expected with respect to DA, *and* the game does not become especially complex.

On the other hand, the switch from DA to CADA is not costless. The following Remark clarifies the significance of the fact that CADA is strategy-proof with respect to the ranking of schools. To understand it, observe that, in any CADA equilibrium, there is a (generically one) "worst" school such that any student that applies there is accepted. This observation was necessary to make Remark 2 more precise.

Remark 3 *1) In any CADA equilibrium, the student's target determines the most-preferred object for which the student has a positive assignment probability, except possibly for the "worst" school.*

2) Define DA with First-Choice Priority (DA-FCP) as: students reveal ordinal preferences; two independent tie-break lotteries L_1 and L_2 are run; a DA procedure follows with the following tie-breakers: a) whether the school pointed to is first choice, b) between any two students pointing to the same first choice, L_1, and c) for the remaining ties, L_2. It can be shown that for any m and $\vec{\eta}$, both DA-FCP and CADA induce the same random assignments. Also, DA-FCP is not strategy-proof.

That this remark is true can be seen along the lines of Abdulkadiroğlu, Che and Yasuda (2008). In equilibrium, the revealed ranking among overdemanded schools that are preferred to any underdemanded school does not affect the random assignment. If we constrain the strategy space to equal the one of DA and BM, a mechanism equivalent to CADA is no longer strategy-proof.

To see again that ranking revelation plays a less important role, consider an extension of CADA, named CADA-k, where students are allowed to name k ordered target schools. The subsequent DA procedure establishes the following tie-breakers: 1) whether the student set the school as her first target, 2) whether the student set the school as her second target, ... k) whether the student set the school as her k-th target, and k+1) a fair independent lottery number for each target level, plus another independent lottery to break ties among students who point to the same school while not targeting it. The next Proposition states that CADA-k and BM are *ex ante* equivalent for k high enough.

Proposition 4 *Fix m and $\vec{\eta}$. Then, with a continuum of students and no school priorities, both CADA-k and BM induce the same random assignments if $k \geq J - 2$.*

Proof. See the second part of the appendix. ■

When CADA is sufficiently extended, the ranking revelation part of each student's strategy has no importance whatsoever in the resulting random assignment. This extended CADA becomes the Boston Mechanism.[39]

The idea is that, if students' revealed ordinal preferences did not play any role in CADA-J assignment, CADA-J would be equivalent to BM: choosing a school as one's j-th target has the same effect in CADA-J assignment as ranking it j-th has in BM assignment. In CADA-J, a student may have a chance of being accepted at her j-th target if: a) previously targeted schools reject the student, and b) the school has previously accommodated all students who targeted the school higher and there are still some slots left. It is not hard to see that BM works analogously. Therefore, the key argument is that ranking revelation does not play any role in any CADA-J *equilibrium* assignment. The CADA-$(J-1)$ case is just an obvious extension: the last target is a redundant element in each student's strategy. The CADA-$(J-2)$ case relies on the fact that students sincerely rank their two last-ranked schools in any BM equilibrium.

This equivalence may not hold when schools have priorities over students. CADA, as proposed by Abdulkadiroğlu, Che and Yasuda (2008), puts school priorities ahead of target-driven priorities in the tie-breaker hierarchy, whereas BM gives higher importance to ranking-driven priorities than to school priorities.

CADA is a compromise between DA and BM, and it is a very appealing way to use preference intensities when assigning probabilities. Nevertheless, the results stated above illustrate that a trade-off between strategy-proofness and *ex ante* efficiency gains persists. CADA does not come for free.

While I have shown that some extra *ex ante* efficiency properties were already present in BM, I do not claim that BM outperforms CADA in all cases. Indeed, Abdulkadiroğlu, Che and Yasuda (2008) illustrate via example that CADA can be *ex ante* efficient in scenarios where BM is not, and vice versa.[40]

For a special case, I do have one analytical comparative result between these two mechanisms. If all students have the same ordinal preferences, and the second-best school is not relatively highly valued for sufficiently many students, then CADA cannot be Pareto-preferred to BM.

[39] In a recent version of their paper, Abdulkadiroğlu, Che and Yasuda (2008) note this fact for the case where all students have the same ordinal preferences and schools have no priorities.

[40] They compare CADA to CADA-2 in a four-school example. CADA-2 turns out to be *ex ante* equivalent to BM, given Proposition 4 in the present paper.

Proposition 5 *Fix $\vec{\eta}$, and let m be such that all students share identical ordinal preferences $o = (1, 2, ..., J)$ and $m(\{v \in V : v_2 \geq \vec{\eta}' \cdot v\}) \leq \eta_2$. Then there exists $q \in Q^{BM}(m, \vec{\eta})$ such that no $\tilde{q} \in Q^{CADA}(m, \vec{\eta})$ ex ante Pareto-dominates q.*

Proof. See the second part of the appendix. ■

After having assigned slots at targeted schools, CADA splits the remaining school capacities evenly randomly among non-placed students, when all of them share identical ordinal preferences. That is an *abysmal* way to assign the remaining slots (in the sense of Proposition 2). All students would do (weakly) better if a BM-like mechanism were applied to assign them. Since any student who is not assigned to her first-ranked option in BM does (weakly) better than she would do in CADA, she can try a rather less conservative first choice in the former mechanism. I prove that this will indeed happen in some BM equilibrium if there is a CADA equilibrium where only school 1 is overdemanded (for which $m(\{v \in V : v_2 \geq \vec{\eta}' \cdot v\}) \leq \eta_2$ is necessary and sufficient). Those who were targeting a safe underdemanded school in any CADA equilibrium and now rank a more preferred school first in the BM equilibrium improve their payoffs.

The following corollary follows from Propositions 4 and 5.

Corollary 2 *With four schools and identical ordinal preferences across students, there exists $q \in Q^{BM}(m, \vec{\eta})$ such that no $\tilde{q} \in Q^{CADA}(m, \vec{\eta})$ ex ante Pareto-dominates q.*

I have performed some simulations with CADA, in order to compare BM to this mechanism. CADA equilibrium computation is quite difficult when weak priorities are included, so comparisons are restricted to the no-priority case.

In general, the figures below suggest that the differences between BM and CADA are small (less than 1%) if there are few schools. Figure 9 illustrates the differences in utilitarian *ex ante* efficiency between the two mechanisms, in percentages. Two main patterns emerge. On the one side, BM tends to be more efficient when the correlation among students' vNM valuations is high, while CADA is more efficient in the opposite case. When correlation is high, students tend to have very similar ordinal preferences. In CADA, students who do not get accepted at their targeted schools are near-evenly assigned to the remaining slots. BM can improve over that, since students can strategize over their second, third and so forth ranked schools, so that cardinal utilities influence the assignment of remaining slots.

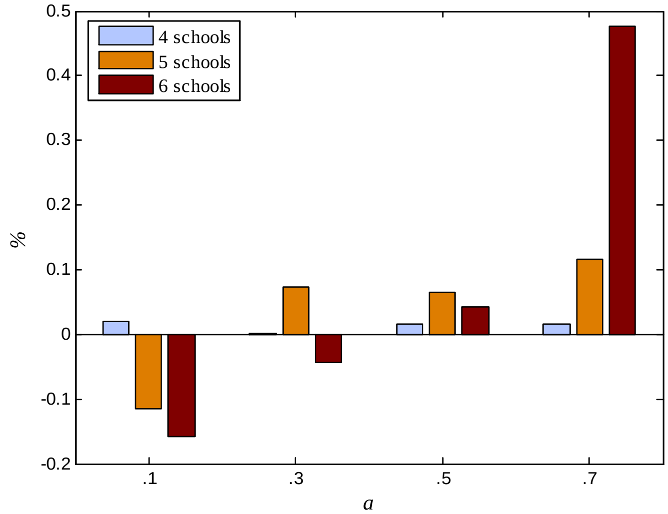

Fig. 9: % Welfare gain BM-CADA, 20 students/school.

On the other side, differences between BM and CADA seem to be amplified when the number of schools increase. BM and CADA are *ex ante* equivalent when there are less than three underdemanded schools. With four schools, differences occur only when there are exactly three underdemanded schools. The number of possible differing cases increases when the number of schools becomes larger.

One can observe that these differences are small as compared to the differences between BM and DA, by noting the scale of the respective graphs. I conjecture that the former differences would be larger if the number of schools were higher. A 75-draw simulation with seven schools shows that BM outperforms CADA by .66%, in 20 students/school scenarios when $a = .7$.[41]

[41] Since the number of available strategies per student is the factorial of the number of schools, an increase of one school has a tremendous marginal computational cost.

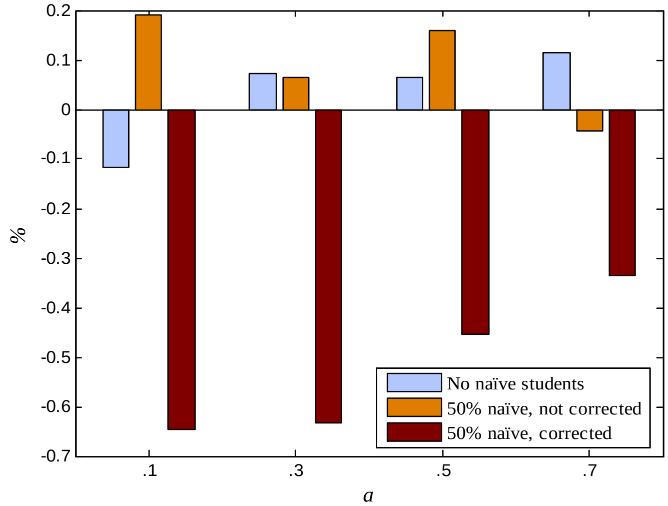

Fig. 10: % Welfare gain BM-CADA, 5 schools, 20 stud./school.

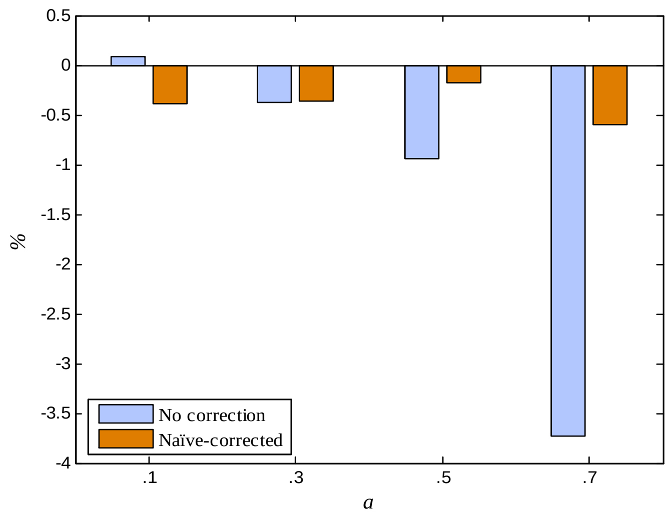

Fig. 11: Naïve students' % welfare differences BM-CADA, 5 schools, 20 stud./sch.

Figure 10 may suggest an interesting alternative pattern with naïve students, although differences are still small with six schools or fewer. With no naïve students, BM is more efficient than CADA provided the correlation among students' valuations is high. With a substantial portion of naïve students, the opposite happens. If a corrective device is applied to BM to protect naïve students, an efficiency loss arises. This loss decreases as the correlation among students' valuations becomes larger.

Figure 11 illustrates once again that the protective device is effective in saving a substantial portion of naïve students' losses in BM. When the correlation parameter is .7, naïve students' welfare losses with respect to CADA are reduced from around 3.75% with no correction to scarcely more that .5% if the correction is implemented.

8.2 Proofs

Proof. Proposition 1.

With a continuum of students, Abdulkadiroğlu, Che and Yasuda (2008) have shown that any assignment induced by DA can be characterized by a vector of cutoffs $(c_1, ..., c_J)$, one for each school. A student is assigned to a school she applies to (after having been rejected at other preferred schools) if and only if her assigned lottery number is lower than the school's cutoff. The set of all possible lottery numbers is the interval $[0, 1]$, and these are uniformly assigned.

Consider a strategy profile $r : V \to \Pi(S)$. Write $\mu_r(c, j)$ for the measure of student types who *apply* for school j (meaning that they were rejected at any other school that was put in a higher ranking position) whose highest cutoff they have faced in previous rejections is c. I adopt the notation $\mu_r(0, j)$ for the measure of student types who situate school j in first position. Without loss of generality (WLOG), arrange schools in increasing cutoff order. The following recursive equations apply:

$$\begin{aligned} c_1 &= \frac{\eta_1}{\mu_r(0,1)} \\ \eta_j &= \mu_r(0,j)c_j + \sum_{i<j} \mu_r(c_i,j)(c_j - c_i), \;\; j > 1 \end{aligned}$$

In the PS mechanism (the cake-eating algorithm), write t_j for the "cake-end" time for school j. It represents the moment in which slots at school j become unavailable (they have been completely "eaten"). Set an eating speed of 1 for each student. Consider again the same strategy profile r. Denote by $\tilde{\mu}_r(t, j)$ the measure of student types who at some point "eat" from school j given that their latest (declaredly) preferred school to become unavailable does so at time t. WLOG, arrange schools in increasing "cake-end" time order. Then "cake-end" times can be recursively calculated from:

$$\begin{aligned} t_1 &= \frac{\eta_1}{\tilde{\mu}_r(0,1)} \\ \eta_j &= \tilde{\mu}_r(0,j)t_j + \sum_{i<j} \tilde{\mu}_r(t_i,j)(t_j - t_i), \;\; j > 1 \end{aligned}$$

It is clear that the vector of cutoffs equals the vector of "cake-end" times. Now, if a v-type student sets a ranking $(s_1, ..., s_J)$ according to r, then $q_{s_1}^{DA}(v) = c_{s_1}$ and $q_{s_j}^{DA}(v) = \max\{0, c_{s_j} - \max_{i<j} c_{s_i}\}$ for $j > 1$. But also $q_{s_1}^{PS}(v) = t_{s_1}$ and $q_{s_j}^{PS}(v) = \max\{0, t_{s_j} - \max_{i<j} t_{s_i}\}$ for $j > 1$. Therefore, the mechanisms are *ex ante* equivalent. ■

Proof. Theorem 1.

Define the relation $\triangleright$ between schools given $(q, m, \vec{\eta})$ as

$$i \triangleright j \iff \exists V_{ij} \subset V : m(V_{ij}) > 0,\ \forall v \in V_{ij} \quad v_i > v_j, q_j(v) > 0$$

With a finite number of students, Kojima and Manea (2007) show that the ordinal random assignment q is ordinally efficient at $(m, \vec{\eta})$ if and only if the associated relation $\triangleright$ is acyclical.[42] The easy extension of this result to a continuum of students is omitted.

The next step is to show that acyclicity implies, and is implied by, *ex post* efficiency. That acyclicity implies *ex post* efficiency is clear. An *ex post* Pareto-improvement consists of a (finite) trading cycle. Namely, student 1 prefers the assignment of student 2, who prefers the assignment of student 3,... who prefers the assignment of student k, who prefers the assignment of student 1. Trading cycles occur for a zero measure $m \times l$ of students, due to the acyclicity of $\triangleright$.

So the proof ends by showing that *ex post* efficiency implies acyclicity. Here the continuum of students play its role. The lottery has been designed to give a constant conditional probability measure m of student types for any measurable subset of lottery elements. Let $\triangleright$ have a cycle $1 \triangleright 2 \triangleright ... \triangleright s \triangleright 1$. Let $V_{12} = \{v \in V : v_1 > v_2, q_2(v) > 0\}$. We can also define $V_{23}, ..., V_{s1}$ accordingly. Since all these sets have positive measure, a positive measure of lottery numbers give student types in any V_{ij} an assignment to j. By the way the lottery is constructed, there will be a positive measure $m \times l$ of students receiving the proper lottery numbers. Thus, one can construct an *ex post* improving trading cycle. ■

Proof. Proposition 2.

Define the Uniform Mechanism (UM) as the one that randomly assigns slots evenly among students, regardless of their types. DA and UM are *ex ante* equivalent when all students have identical ordinal preferences. Proposition 2 is a corollary from the following Theorem. ■

Theorem 2 *With a continuum of students, UM is an abysmal mechanism in all environments.*

[42] They also add a non-wastefulness condition, which is not needed here in absence of outside options.

Proof. It is a generalization of Theorem 6 in Abdulkadiroğlu, Che and Yasuda (2008). Let Σ be the strategy space in some (anonymous) mechanism M, and let σ be any element in Σ. Denote the measure of students finally playing strategy σ as μ_σ, and the assignment probability to school j for a student who plays σ as $\rho_j(\sigma)$. For any school j, feasibility constraints imply $\sum_{\sigma\in\Sigma}\mu_\sigma\rho_j(\sigma)=\eta_j$. Let a generic mixed strategy be denoted as $\{\nu_\sigma\}_{\sigma\in\Sigma}$. A student who plays a mixed strategy $\nu_\sigma=\mu_\sigma$ for any $\sigma\in\Sigma$ obtains an assignment probability η_j for school j. But this is exactly the random assignment one obtains in UM. So any student can do at least as well as in UM no matter what other students' chosen strategies are. ■

Proof. Proposition 3 (conditions 1 to 3).

1) *ex post efficiency*: In light of Theorem 1, one just needs to prove that $\rhd$ is acyclic. In BM, I define a school s as *round-t overdemanded* ($s\in O_t$) if it was not round-($t-1$) overdemanded and it has fulfilled all its slots at the end of round t (to complete the definition, $O_0=\varnothing$). It is *round-t underdemanded* ($s\in U_t$) if $s\notin O_1\cup...\cup O_t$.

First, I argue that if there is a cycle $s_1\rhd s_2\rhd...\rhd s_k\rhd s_1$, then there are no $s_a,s_b\in\{s_1,...,s_k\}$ and round t such that $s_a\in O_t$ and $s_b\in U_t$. For any $s_b\in U_t$, there are no $v-$type students who strictly prefer s_b to all $s_a\in O_t$ and rank some s_a ahead of s_b. She would be suboptimally choosing her strategy in that case, since s_b is a sure placement in round t and earlier rounds. Thus, for any pair $s_a\in O_t$ and $s_b\in U_t$, $s_b\ntriangleright s_a$, and the cycle cannot be completed.

Thus, if there is any cycle $s_1\rhd s_2\rhd...\rhd s_k\rhd s_1$, all its components must belong to the same set O_t, for some round t. I finish the proof by showing that this case is not possible either. Due to Round-wise Tie-Breakers, for each school $s_a\in O_t$ all students ranking s_a in $t-$th position have, upon rejection in previous rounds, the same probability of being accepted at s_a, namely ϕ_{at}.

Also, conditional of being rejected at round t, assignment probabilities in further rounds do not depend on the choice s_a, due again to RTB. Thus, if $s_a,s_b\in O_t$ and $s_a\rhd s_b$, it must be the case that $\phi_{at}<\phi_{bt}$. Suppose $s_a,s_b\in O_t$ and $s_a\rhd s_b$. Both schools were underdemanded in previous rounds, so it would be suboptimal for any student who ranked either s_a or s_b in position $\tau<t$ to pick the least preferred one among the two. Thus a positively measured set of student types picked s_b in t-th position when preferring s_a. It must be the case that the gain in probabilities compensates for the loss in vNM valuations (recall that the picked school does not affect assignment probabilities in further rounds upon rejection in round t), thus $\phi_{at}<\phi_{bt}$. Then, if a cycle $s_1\rhd s_2\rhd...\rhd s_k\rhd s_1$ exists, we must have $\phi_{1t}<\phi_{2t}<...<\phi_{kt}<\phi_{1t}$, which is a contradiction.

2) *ex ante efficiency within the set of overdemanded schools*: Let $r^*:V\rightarrow\Pi(S)$ be a Nash

equilibrium in the BM game, given m and $\vec{\eta}$. Let $S^*_{r^*}$ denote the set of overdemanded schools. For any $s \in S^*_{r^*}$, any student type who did not put s first in her declared ranking has no chance to get a slot there, since its slots are assigned in the very first round.

A v-type who put $s \in S^*_{r^*}$ first in her ranking obtains the following expected payoff:

$$\pi_{r^*}(v) = q_s^{r^*}(v) \cdot v_s + \left(1 - q_s^{r^*}(v)\right) \cdot v_w$$

where v_w is a convex combination of v-type's vNM valuations for underdemanded schools given r^*. Such a v-type student prefers her first-choice school s to any underdemanded school (otherwise she could get a sure slot at that underdemanded school by ranking it first). It is clear then that $v_s \geq v_w$. RTB implies that v_w is unaffected by the specific overdemanded school that is ranked first.

Given that, this type is choosing $r^*_1(v) \in S^*_{r^*}$ so as to solve $\max_{s \in S^*_{r^*}} q_s^{r^*}(v)(v_s - v_w)$. For each overdemanded school s, $q_s^{r^*}(v) = \eta_s / m(\{\tilde{v} \in V : r^*_1(\tilde{v}) = s\})$. Notice that it does not depend on v.

Since all types who did not rank any $s \in S^*_{r^*}$ first have zero assignment probabilities for each of these schools, any reassignment that keeps probabilities unchanged in $S \backslash S^*_{r^*}$ still obtains zero probabilities in $S^*_{r^*}$ for these types. Hence I can safely ignore them.

Construct an artificial pseudomarket encompassing all overdemanded schools with their own capacities and an unlimited outside option w which is valued v_w for each v-type student. Only students who ranked an overdemanded school first in the BM equilibrium are considered, and each student is endowed with one unit of budget. In this pseudomarket, students buy units of allocation probabilities, and there is an equilibrium price $p^*_s \geq 1$ associated to each overdemanded school. WLOG (see Hylland and Zeckhauser, 1979), I can set p^*_w, the price of the unlimited good, to zero.

Except in case of indifference (which happens for a measure 0 of student types), each v-type student optimally chooses to spend all her endowment in buying probability units for the overdemanded school $d^*(v)$ that solves $\max_{s \in S^*_{r^*}} (v_s - v_w)/p^*_s$. Since each p^*_s is determined so as to clear that school's market, I obtain that for each $s \in S^*_{r^*}$, $p^*_s = m(\{\tilde{v} \in V : d^*(\tilde{v}) = s\})/\eta_s$. It is easy to see that this artificial pseudomarket equilibrium replicates the BM equilibrium decisions $r^*_1(\cdot)$ for those student types who put an overdemanded school first in their rankings. Since a pseudomarket equilibrium is *ex ante* efficient (Hylland and Zeckhauser, 1979), no feasible rearrangement of probabilities in $S^*_{r^*}$ can possibly obtain *ex ante* Pareto-improvements. Hence the random assignment from the BM equilibrium r^* is *ex ante* efficient within $S^*_{r^*}$.

3) *ex ante efficiency if there is only one underdemanded school*: it follows from the proof of the previous condition. It is easy to see that the "outside option" w is the underdemanded school, so it

does not need to be artificially constructed. Thus BM and the Pseudomarket yield identical random assignments. ■

Proof. Lemma 1.

For $s = 1$, the proof is similar to the proof of Proposition 3 (condition 2), where w in now understood as the unique underdemanded school. This argument shows that both BM and CADA reach the same equilibrium outcomes as the Pseudomarket mechanism proposed by Hylland and Zeckhauser (1979). So consider $s = 2$ in the rest of the proof.

My aim is to show that, following the lines of the previous proof, v_w in equilibrium is the same in BM as in CADA. Consider a BM-equilibrium r^* with two underdemanded schools i and j. Given $r_1^*(v)$, a v-type student chooses $r_2^*(v), ..., r_J^*(v)$ so as to maximize v_w. With only two underdemanded schools, one way to do so is letting $r_2^*(v) = \arg\max_{h \in \{i,j\}} v_h$ (just because BM is strategy-proof when $J = 2$). Other optimal choices are possible but they keep payoffs unchanged for all student types. In CADA with equilibrium τ^* where i and j are underdemanded, v_w is determined by a (strategy-proof) DA procedure between i and j. Since BM and DA are both equivalent procedures when $J = 2$, v_w will be the same in both cases.

Therefore, r_1^* (first choices in BM-equilibrium) and τ^* (the CADA equilibrium targets) are the same, since they solve the same maximization program for each v-type: either $\max_{s \in S \setminus \{i,j\}} (v_s - v_w)\eta_s / m(\{\tilde{v} : r_1^*(\tilde{v}) = s\})$ (use τ^*instead of r_1^* in CADA), or choose i (or j) if i (or j) $= \arg\max_{s \in S} v_s$.
■

Proof. Proposition 4.

In the CADA-J game, a (pure) strategy profile is a function $\vec{\tau} = (\tau_1, ..., \tau_J) : V \to \Pi(S)$ of first, second,... and J-th targets. In CADA-J, define a school j as *target-t overdemanded* ($j \in O_t$) given $\vec{\tau}$, m and $\vec{\eta}$ if it is not target-$(t-1)$ overdemanded and fills all its slots with students who have targeted j in t-th position or better (to complete the definition, $O_0 = \varnothing$). A school j is *target-t underdemanded* ($j \in U_t$) if $j \notin O_1 \cup ... \cup O_t$.

Consider an equilibrium $\vec{\tau}^*$ of the CADA-J game under m and $\vec{\eta}$. Any such equilibrium satisfies $\#(O_1 \cup ... \cup O_t) \geq t$ for any $t \in \{1, ..., J\}$. To see this, suppose $O_t = \varnothing$, for some $t \in \{1, ..., J\}$. It must be the case then that a positive measure of student types have targeted elements in $O_1 \cup ... \cup O_{t-1}$ in $t-$th position. These types must have targeted overdemanded schools for any $\tau < t$, since otherwise they would have obtained sure assignment before t. Hence $\#(O_1 \cup ... \cup O_t) \geq t$.

For each t, among the schools in U_{t-1}, any student type whose more preferred school s_u belongs to

U_t optimally signals $\tau_t = s_u$. If $s \in U_{t+1}$, the student might as well signal $\tau_t = s_{n/a} \in O_1 \cup ... \cup O_{t-1}$ with no loss of payoff (as long as s_u is properly targeted further), but this does not change the random assignment, since the student has no chance at $s_{n/a}$ and will readily apply to s_u when it is still target underdemanded. Besides, this student might list some target-t overdemanded schools ahead of s_u in her ordinal preferences, but again this does not change the random assignment since the student has no chance of acceptance at such schools. If a student targets a school $s_o \in O_t$ in t-th position, it must be the case that she prefers it to any target-t underdemanded school, so she will never apply to any of the latter before applying to s_o. This student might list some target-t overdemanded schools ahead of s_o in her ordinal preferences, but the student has no chance of acceptance there and will eventually apply to s_o.[43]

Given the previous paragraph and $\#(O_1 \cup ... \cup O_J) = J$, the DA procedure in CADA-J ignores students' revealed ordinal preferences and just uses students' targets in equilibrium.

Since no assignment given $\vec{\tau}^*$, m and $\vec{\eta}$ uses the information on students' ranking of objects, each assignment is equivalent to one from BM given $r = \vec{\tau}^*$, m and $\vec{\eta}$. This is because $\vec{\tau}^*$ generates endogenous hierarchies of DA priorities that have the same effects as definite acceptances do in BM. The targets in CADA-J play the same role as the ranking does in BM. It follows that $\vec{\tau}^*$ is also an equilibrium in the BM game.

The equivalence result obviously holds between CADA-$(J-1)$ and BM since both the last target and the last-ranked school are redundant elements in each student's strategy. That the equivalence exists between CADA-$(J-2)$ and BM can be understood from the fact that any equilibrium strategy in the BM game honestly ranks the two last-ranked schools (just as BM is strategy-proof when $J = 2$). In CADA-$(J-2)$, truthful rankings determine the assignment to untargeted schools with some capacity left after targeters have been accommodated. In a BM-equilibrium r^* there are at most two schools with positive unassigned capacity at round $J-1$, because the property $\#(O_1 \cup ... \cup O_t) \geq t$ holds also for BM. A CADA-$(J-2)$ equilibrium has a similar assignment property in equilibrium: after all target priorities have been used in assigning school slots, only two schools (at most) have positive remaining capacity. Since BM and DA work identically with only two schools, for each BM-equilibrium there is a CADA-$(J-2)$ equilibrium that achieves the same random assignment, and vice versa. ■

[43] This part of the proof is inspired by the proof of Lemma 3 in Abdulkadiroğlu, Che and Yasuda (2008).

Proof. Proposition 5.

First, I show that there exists a target equilibrium τ^* in CADA such that only school 1 is overdemanded, if and only if $m(\{v \in V : v_2 \geq \vec{\eta}' \cdot v\}) \leq \eta_2$. Consider any target profile τ such that only school 1 is overdemanded, and no one targets any school $s > 2$. Let $\mu_2^\tau \leq \eta_2$ denote the measure of student types targeting school 2. Targeting some school $s > 2$ is never a best response to τ, so a v-type student either secures v_2 by targeting school 2, or she targets school 1 obtaining:

$$\begin{aligned} & \frac{\eta_1}{1-\mu_2^\tau} v_1 + \left(1 - \frac{\eta_1}{1-\mu_2^\tau}\right) \frac{(\eta_2 - \mu_2^\tau) v_2 + \sum_{j>2} \eta_j v_j}{1 - \mu_2^\tau - \eta_1} \\ = \; & \frac{\vec{\eta}' \cdot v - \mu_2^\tau v_2}{1 - \mu_2^\tau} \end{aligned}$$

This is higher than v_2 if and only if $v_2 < \vec{\eta}' \cdot v$. So the student's best response is determined by this inequality, which does not depend on μ_2^τ. The (essentially) unique equilibrium where only school 1 is overdemanded is characterized by all types such that $v_2 < \vec{\eta}' \cdot v$ targeting school 1, and those in which $v_2 > \vec{\eta}' \cdot v$ targeting school 2. Such an equilibrium exists if and only if $m(\{v \in V : v_2 \geq \vec{\eta}' \cdot v\}) \leq \eta_2$, which keeps school 2 underdemanded. Let $\mu_2^{\tau^*} \equiv m(\{v \in V : v_2 \geq \vec{\eta}' \cdot v\})$.

In a generic BM strategy profile r, let μ_π denote the measure of student types who choose $\pi \in \Pi(S)$, and μ_j the measure of them whose first-choice school is j. Let $\vec{\mu} \equiv (\mu_\pi)_{\pi \in \Pi(S)}$. Finally, define the set $\Theta = \{\vec{\mu} \in \Delta^{\#\Pi(S)-1} : \mu_2 \leq \mu_2^{\tau^*}, \mu_s = 0 \; \forall s > 2\}$. Note that Θ is compact.

Define the function $\beta : \Delta^{\#\Pi(S)-1} \to \Delta^{\#\Pi(S)-1}$ as the result of a best-response profile at BM. That is, if $\vec{\mu}$ is given by r, $\beta(\vec{\mu})$ is given by the profile of best responses to r. It is a function since best responses are singletons almost everywhere. It is also continuous under the atomlessness assumption. I show that $\beta(\Theta) \subset \Theta$.

Consider any strategy profile r in BM such that its implied $\vec{\mu}$ belongs to Θ (and thus $\mu_2 \leq \mu_2^{\tau^*}$). If CADA is played and such a measure μ_2 of students target school 2 (and the rest point to school 1), a v-type student who targets school 1 and is rejected there receives

$$v_{CADA}(\mu_2) \equiv \frac{(\eta_2 - \mu_2) v_2 + \sum_{j>2} \eta_j v_j}{1 - \mu_2 - \eta_1}$$

This is higher than $v_{CADA}(\mu_2^{\tau^*})$ (strictly so if $\mu_2^{\tau^*} > \mu_2$). Now, if that student best-responds to r by a ranking that puts school 1 first and she is rejected there, she obtains $v_{BM}(\mu_2)$. Following Proposition 2, $v_{BM}(\mu_2) \geq v_{CADA}(\mu_2)$, since she cannot do worse than in an evenly random split of remaining slots.

This implies that the measure of student types ranking school 2 as part of a best response in BM is (weakly) lower than the measure of student types best responding at CADA by targeting that school

$(\mu_2^{\tau^*})$. To see this, note that the payoff from ranking school 1 first in BM is (weakly) higher than the payoff from targeting the same school in CADA, regardless the student type. Hence $\beta(\Theta) \subset \Theta$. Since $\beta(\cdot)$ is continuous and Θ is compact, Brower's fixed-point theorem applies. Therefore, there exists a BM equilibrium r^* such that its associated $\vec{\mu}^*$ belongs to Θ. Let μ_2^* denote the measure of student types ranking school 2 first in this equilibrium.

If $\mu_2^{\tau^*} > \mu_2^*$, then there is a strictly positive measure of student types that were targeting school 2 at the CADA equilibrium τ^* and now rank school 1 first at r^*. These students are better-off in BM than in CADA. If $\mu_2^{\tau^*} = \mu_2^*$, no student is worse-off in BM than in CADA, since $v_{BM}(\mu_2^{\tau^*}) \geq v_{CADA}(\mu_2^{\tau^*})$ for every student type. If we consider another target equilibrium τ', we always find a positive measure of student types who improve their payoffs if the mechanism is switched to BM and the equilibrium r^* is played (for instance, students targeting some school $s > 2$ at τ'). ■